Open maps between Knaster continua

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Abstract

We investigate the set of open maps from one Knaster continuum to another. A structure theorem for the semigroup of open induced maps on a Knaster continuum is obtained. Homeomorphisms which are not induced are constructed, and it is shown that the induced open maps are dense in the space of open maps between two Knaster continua. Results about the structure of the semigroup of open maps on a Knaster continuum are obtained and two questions about the structure are posed.

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1 Introduction

Following the notation of J.W. Rogers [10], for each positive integer n let $w_n : I = [0,1] \to I$ denote the mapping which is 0 at i/n for i even, 1 at i/n for i odd, and linear in between, that is,

$$w_n(x) = \begin{cases} nx - i & \text{if } i \text{ is even and } 0 \leq \frac{i}{n} \leq x \leq \frac{i+1}{n} \leq 1 \\ i + 1 - nx & \text{if } i \text{ is odd and } 0 < \frac{i}{n} \leq x \leq \frac{i+1}{n} \leq 1 \end{cases}$$

The map w_n is called **the standard map of degree n**, and the set of all the maps w_n is denoted by $\mathcal{W}$. As noted in [6], the composition $w_n w_m$ of two standard maps is the standard map w_{mn}, and so $\mathcal{W}$ is a semigroup of mappings on I which is naturally isomorphic with the multiplicative semigroup of positive integers under the function $w_n \to deg(w_n) = n$.

If π is any sequence of positive integers and K_π denotes the inverse limit $\varprojlim\{I_k, \pi_k^{k+1}\}$, where $I_k = I$ and $\pi_k^{k+1} = w_{\pi(k)}$, then K_π is an indecomposable continuum (compact connected metric space) except in the case that $\pi(i) = 1$ for all but finitely many i [9].

If the sequence π is a constant sequence $\pi(k) = n$, then we denote K_π by K_n. The continuum K_2 is the well-known 'bucket handle' described in the 1920's by Knaster as an intersection of disks in the plane. We refer to K_π as a **Knaster continuum** and denote the set of all of homeomorphism classes of Knaster continua by $\mathbb{K}$.

Knaster continua have been studied by many authors, including J. W. Rogers [10], and W. Debski [6].

In [10], Rogers shows that each indecomposable metric continuum can be mapped continuously onto any Knaster continuum, and that any inverse limit $\varprojlim\{I_i, f_i^{i+1}\}$ is (homeomorphic with) a Knaster continuum if each map f_i^{i+1} is a limit of open maps in the sup metric.

In [6], Debski provides a complete classification of Knaster continua and shows that there are uncountably many topologically different Knaster continua.

In this paper, we investigate the structure of the open mappings between Knaster continua. If π and ρ are sequences of primes, then $\mathcal{O}_\rho^\pi$ denotes the set, possibly empty, of all open mappings $f : K_\pi \to K_\rho$. In case $\pi = \rho$, $\mathcal{O}_\pi^\pi$ will be written $\mathcal{O}_\pi$. This last set forms a semigroup under composition of functions, since the composition of open maps is open.

Let $\mathbb{P}$ be the set of primes and $\omega = \{0, 1, 2, \cdots, \infty\}$ the set of countable cardinals.

Every sequence π of primes has associated with it an **occurrence function**

$$occ_\pi : \mathbb{P} \to \omega$$

whose value at a prime p is the number of occurrences of p in the sequence π.

Since π is an infinite sequence of primes, either $occ_\pi(p)$ must be ∞ for at least one prime p or $occ_\pi(p)$ must be nonzero for infinitely many primes p. Conversely, given a function $\tau : \mathbb{P} \to \omega$ such that $\tau(p) = \infty$ for some prime p or $\tau(p) > 0$ for infinitely many primes p, we can arrange a sequence π of primes such that $occ_\pi = \tau$.

The semigroup of open mappings on the interval is described in section 2. The structure we find in this semigroup is a key to unlocking the structure of the open induced maps between Knaster continua, which we describe in section 3.

A map $f : K_\pi \to K_\rho$ is said to be an **induced map** provided that there is an increasing sequence of subscripts i_k and maps $f_k : I_{i_k} \to I_k$ so that $\rho_k f = f_k \pi_{i_k}$ for each $k = 1, 2, \cdots$. The sequence is called **a defining sequence** of coordinate maps for f. The set of open induced maps from K_π to K_ρ is denoted by $\mathcal{OI}_\rho^\pi$. In the case $\pi = \rho$, write $\mathcal{OI}_\rho^\pi = \mathcal{OI}_\pi$.

We show that the composition of open induced maps is an open induced map whenever the composition is defined. So the set $\mathcal{OI}_\pi$ is a subsemigroup of the semigroup $\mathcal{O}_\pi$.

We show that an open induced map is determined by any one of its coordinate maps. We obtain a structure theorem for the semigroup $\mathcal{OI}_\pi$ which expresses it as a semi-direct product of some of its subsemigroups.

In section 4, we show how to construct homeomorphisms of Knaster continua which are not induced, and prove that each open mapping between Knaster continua is the uniform limit of open induced mappings.

Before the open maps between K_π and K_ρ can be analyzed, we need to look carefully at the open selfmaps on I.

2 The semigroup of open maps on I

Let $\mathcal{O}$ denote the semigroup of open maps from I to I under composition. We call an element f of $\mathcal{O}$ **order preserving** provided that $f(0) = 0$

and denote the set of all these by $\mathcal{O}^+$. Then $\mathcal{O}^+$ is clearly a subsemigroup of the semigroup $\mathcal{O}$.

The following theorem is proved in [10].

2.1 Theorem *For each $f \in \mathcal{O}$, $f : I \to I$ is a surjection. Further, there is a uniquely determined strictly increasing sequence a_i, $i = 0 \cdots n$, with $a_0 = 0$, $a_n = 1$, such that the restriction of f to $[a_i, a_{i+1}]$ is a homeomorphism to I, for each $i = 0 \cdots n - 1$.*

The **degree** of an open mapping f, $deg(f)$ is defined to be the n that satisfies the above theorem.

Let $\mathcal{H}$ ($\mathcal{H}^+$) denote the group of homeomorphisms (order preserving homeomorphisms) of I. Then $\mathcal{H}$ is the group of units of $\mathcal{O}$ and $\mathcal{H}^+ = \mathcal{H} \cap \mathcal{O}^+$ is the group of units of $\mathcal{O}^+$.

Denote by α the homeomorphism $x \to 1 - x$ on I. Then $\alpha^2 = w_1$, the identity map on I.

2.2 Lemma *Let 1 denote the constant function $x \to 1$ on I. Then for any positive integer n,*

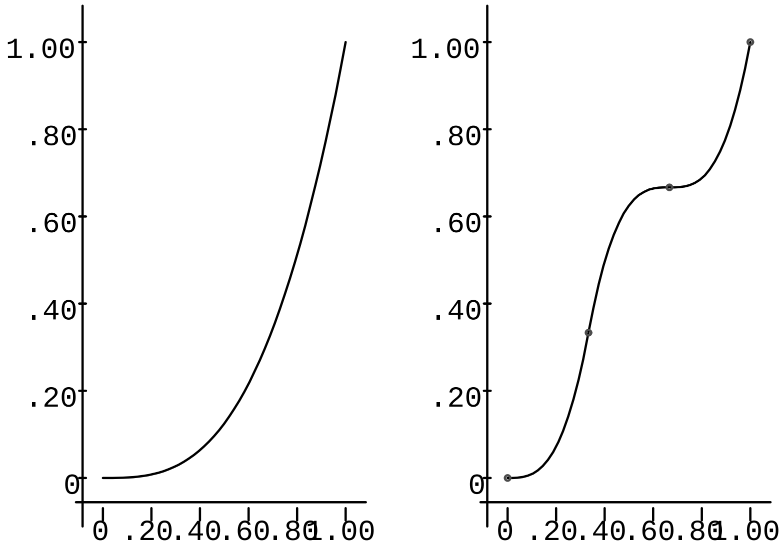
(i) $1 - w_n = \alpha w_n \neq w_n \alpha = w_n$ *when n is even and*

(ii) $1 - w_n = \alpha w_n = w_n \alpha \neq w_n$ *when n is odd.*

The next lemma is found in [10].

2.3 Lemma *If $f : I \to I$ is a continuous function and a_i, $i = 0, \cdots, n$ is an increasing sequence in I on which the values of f alternate between 0 and 1, then there is a continuous function g such that $w_n g = f$. Furthmore, if $a_0 = 0 = f(a_0)$, $a_n = 1$, and the restriction of f to each interval $[a_i, a_{i+1}]$ is 1-1, then g is an order preserving homeomorphism of I.*

If $h \in \mathcal{H}^+$ and $w_n \in \mathcal{W}$, then $f = hw_n \in \mathcal{O}^+$ and $deg(f) = n$, so the the graph of the map g defined in Roger's lemma 2.3 to satisfy $hw_n = w_n g$ is seen to be the union of n scaled copies of the graph of h. (See the diagram below).



So it is reasonable to call g a **multiple of h by n** and to denote g by nh. Also we will denote h by $\frac{1}{n}g$. Note that while nh always exists, $\frac{1}{n}h$ only exists when there is a homeomorphism k such that $nk = h$. This notation is useful for stating the rule for multiplication in $\mathcal{O}$, in the structure theorem below.

2.4 Structure theorem for $\mathcal{O}$ *Each $f \in \mathcal{O}$ can be written uniquely as a product $f = \alpha^i w_n h$, where $i = f(0) \in Z_2$, $deg(f) = n$, and h is in $\mathcal{H}^+$. Furthermore, the rule for multiplication in $\mathcal{O}$ is given by*

$$(\alpha^i w_n h)(\alpha^j w_m g) = \alpha^{i+nj} \ w_{nm} \ m(\alpha^j h \alpha^j) \ g.$$

Proof:

Case 1: f is order-preserving.

Since $f(0) = 0$, f is open, and $deg(f) = n$, we know by Theorem 2.1 that there are numbers a_i with $a_0 = 0 < a_1 < \cdots a_n = 1$ such that $f(a_i) = 0$ if i is even, $f(a_i) = 1$ if i is odd, and f is a homeomorphism on each subinterval $[a_i, a_{i+1}]$. In particular, if $x \notin \{a_0, a_1, \cdots, a_n\}$, then $f(x) \in (0,1)$. By Lemma 2.3, at least one map g satisfying the conditions a), b), and c) exists.

To show that g is unique, suppose $g' \neq g$ is also a such a map. Then, since $f(a_i) = w_n(g(a_i)) = w_n(g'(a_i))$ and $f(a_i) \in \{0,1\}$ for each i, we conclude that g and g' map $\{a_0, \cdots, a_n\}$ into $w_n^{-1}(\{0,1\}) = \{0, 1/n, \cdots, 1\}$. Furthermore, since g and g' are one to one and order preserving, we know that $g(a_i) = i/n = g'(a_i)$ for each i. Since $g \neq g'$, there exist i and x such that $a_i < x < a_{i+1}$ and $g(x) \neq g'(x)$. Then, since $w_n(g(x)) = w_n(g'(x))$, it follows that there is a turning point p of w_n between $g(x)$ and $g'(x)$. Without loss of generality, we may assume that $g(x) < p < g'(x)$. But g and g' are order preserving, so $i/n = g(a_i) < g(x) < p < g'(x) < g'(a_{i+1}) = (i+1)/n$. So p cannot be a turning point of w_n, since i/n and $(i+1)/n$ are consecutive turning points of w_n.

Case 2: f is order-reversing.

Since $f(0) = 1$, note that $\alpha(f(0)) = 0$, so Case 1 applies to αf to factor $\alpha f = w_n h$ uniquely. This yields $f = \alpha\alpha f = \alpha^{f(0)} w_n h$.

To prove that the factorization is unique, suppose that $f = \alpha^j w_m g$, with $g \in \mathcal{H}^+$. Then $i = f(0) = \alpha^j(w_m(g(0))) = \alpha^j(w_m(0)) = \alpha^j(0) = j$. Hence $w_m g = w_n h$, so $m = deg(w_m g) = deg(w_n h) = n$. Finally, by Case 1, $h = g$.

To verify the rule for multiplication,

$$\begin{aligned} (\alpha^i w_n h)(\alpha^j w_m g) &= \alpha^i w_n (\alpha^j \alpha^j) h \alpha^j w_m g \\ &= (\alpha^i w_n \alpha^j)(\alpha^j h \alpha^j w_m) g \end{aligned}$$

Now in the second factor in the last expression, $\alpha^j h \alpha^j w_m$, $\alpha^j h \alpha^j$ is in $\mathcal{H}^+$, so by Case 1

$$(\alpha^i w_n \alpha^j)(\alpha^j h \alpha^j w_m) g = (\alpha^i w_n \alpha^j) w_m (m(\alpha^j h \alpha^j)) g$$

Now each of the last two factors above, $m(\alpha^j h \alpha^j)$ and g, are in $\mathcal{H}^+$ so their composition is in $\mathcal{H}^+$. Further, using Lemma 2.2, and taking cases $j = 0, 1$ and n even or odd, we can write $\alpha^i w_n \alpha^j = \alpha^{i+nj} w_n$. Hence

$$(\alpha^i w_n \alpha^j) w_m (m(\alpha^j h \alpha^j)) g = \alpha^{(i+nj)} w_{nm} m(\alpha^j h \alpha^j) g.$$

This establishes the rule for multiplication. ■

The following corollary is immediate.

2.5 Corollary *The function $deg : \mathcal{O} \to Z^+$ is a homomorphism of the semigroup of open self-maps of I to the semigroup of positive integers under multiplication.*

The next result establishes cancellation properties for $\mathcal{O}$.

2.6 Lemma *Suppose that f, g, and g' are in $\mathcal{O}$. Then*

1) If $deg(f)$ is odd and $fg = fg'$, then $g = g'$.

2) If $deg(f)$ is even, $fg = fg'$, and both g and g' are order preserving or both are order reversing, then $g = g'$.

3) If $gf = g'f$, then $g = g'$.

Proof: Whether the assumption is $fg = fg'$ or $gf = g'f$, it follows by Corollary 2.5, that $deg(g) = deg(g')$. By the Structure Theorem 2.4, there are nonnegative integers m, n, i, j, l and homeomorphisms h, k, and k' in $\mathcal{H}^+$ so that $f = w_m \alpha^i h$, $g = w_n \alpha^j k$, and $g' = w_n \alpha^l k'$.

Invoking the multiplication rule from Theorem 2.4, we have

$$(**)\alpha^{i+mj}\ w_{mn}\ n(\alpha^j h\alpha^j)k = fg = fg' = \alpha^{i+ml}\ w_{mn}\ n(\alpha^l h\alpha^l)k'.$$

Thus, by the uniqueness $(i+mj)\ mod2 = (i+ml)\ mod2$, hence $mj\ mod2 = ml\ mod2$.
Proof of 1): If m is odd, then $j=l$ and (**) becomes

$$(**)\alpha^{i+mj}\ w_{mn}\ n(\alpha^j h\alpha^j)k = fg = fg' = \alpha^{i+mj}\ w_{mn}\ n(\alpha^j h\alpha^j)k'.$$

Now we conclude from uniqueness that $n(\alpha^j h\alpha^j)k = n(\alpha^j h\alpha^j)k'$. Since $n(\alpha^j h\alpha^j)$ is a homeomorphism, $k=k'$. So $g=\alpha^j w_n k = \alpha^k w_n k' = g'$.

Proof of 2): If m is even, and g and g' are both order preserving or both order reversing, then $j=l=0$ or $j=l=1$. In either case $j=l$ and so we get from (**) that $w_{mn}\ n(\alpha^j h\alpha^j)k = w_{mn}\ n(\alpha^j h\alpha^j)k'$. As before, we get $g=g'$.

Proof of 3): This argument proceeds similarly to the above, except that no cases are needed. ∎

A semigroup S is **left cancellative** provided that for all $x,y,z\in S$, $xy=xz$ implies $y=z$. Right cancellative is defined similarly. S is **cancellative** if it is both left and right cancellative.

2.7 Corollary *The semigroup $\mathcal{O}^+$ is cancellative. The semigroup $\mathcal{O}$ is right cancellative, but not left cancellative.*

Proof: Lemma 2.6 shows that $\mathcal{O}^+$ is cancellative, and $\mathcal{O}$ is right cancellative. To see that $\mathcal{O}$ is not cancellative, note that $w_2\alpha = w_2$, but α is not the identity. ∎

Generally speaking, a cancellative semigroup need not be embeddable into a group [4]. However, we show in Corollary 3.12 that there is a Knaster continuum whose group of homeomorphisms contains a naturally embedded copy of $\mathcal{O}^+$.

The semigroup $\mathcal{O}$ is also not commutative, although the subsemigroup of standard maps $\mathcal{W}$ is commutative. In fact, we have the following theorem.

2.8 Theorem *An open mapping $f: I\to I$ is a standard open mapping if and only if it commutes with w_2.*

Proof: Suppose $fw_2 = w_2 f$. Then $f(0) = f(w_2(0)) = w_2(f(0)) = 0$, since $f(0)\in\{0,1\}$. Thus by Theorem 2.4, $f = w_m h$, where $deg(f) = m$ and $h\in\mathcal{H}^+$. Using the rule for multiplication in Theorem 2.4, we see that $w_{2m}h = w_2 f = fw_2 = w_{2m}(2h)$. So by the uniqueness, we have $h = 2h$. But then $h = \lim_{n\to\infty} 2^n h = w_1$. ∎

2.9 Corollary *The semigroup $\mathcal{W}$ is a maximal commutative subsemigroup of $\mathcal{O}$.*

Proof: Any $f\in\mathcal{O}$ which commutes with each standard map must be a standard map by the above theorem. ∎

3 Open Induced Maps between Knaster Continua

Recall that a map $f : K_\pi \to K_\rho$ is induced by the sequence of indices i_k and maps $f_k\colon I_{i_k} \to I_k$ if $\rho_k f = f_k \pi_k$ for all positive integers k.

This means that in figure 1, the trapezoid with sides f_k and f commutes and the trapezoid with sides f_l and f commutes. It follows from this definition that for each k, l with $k < l$, the trapezoid with sides f_k and f_l commutes, that is, $\rho^l_k f_l = f_k \pi^{i_l}_{i_k}$. To see this, note first that

$$f_k \pi^{i_l}_{i_k} \pi_{i_l} = f_k \pi_{i_l} = \rho_k f = \rho^l_k \rho_l f = \rho^l_k f_l \pi_{i_l}.$$

But π_{i_l} is a surjection and so can be cancelled on the right to establish the claim.

Figure 1: f is induced by sequences i_k and f_k

$$\begin{tikzcd}
 & & & I_{i_k} \arrow[dlll, "f_k"'] & I_{i_l} \arrow[l, "\pi^{i_l}_{i_k}"'] \arrow[dlll, "f_l"'] & \cdots & K_\pi \arrow[d, "f"] \arrow[lll, bend right, "\pi_{i_k}"'] \\
I_k & I_l \arrow[l, "\rho^l_k"'] & \cdots & I_{i_k} & I_{i_l} \arrow[l, "\rho^{i_l}_{i_k}"'] & \cdots & K_\rho \arrow[llllll, bend left, "\rho_k"]
\end{tikzcd}$$

3.1 Lemma *A map $f\colon K_\pi \to K_\rho$, induced by sequences i_k and f_k, is open if and only if all of the maps f_k are open.*

Proof: Suppose f is open. Since all of the bonding maps ρ^i_j are open, the projections ρ_k are open. So $\rho_k f$ is open for each k. But $\rho_k f = f_k \pi_{i_k}$ and since π_{i_k} is open, it follows that f_k is open for all k.

Now suppose all of the maps f_k are open. Let U be a basic open set in K_π. Then there is a natural number i_j and an open set $V \subset I_{i_j}$ such that $U = \pi^{-1}_{i_j}(V)$. We claim that $f(U) = \rho^{-1}_j f_j \pi_{i_j}(U) = \rho^{-1}_j f_j(V)$, which is clearly open in K_ρ.

Proof of the claim: Suppose that $y \in f(U)$. Then there is a point $x \in U$ such that $f(x) = y$.

Now $\rho_j f(x) = f_j \pi_{i_j}(x)$, by the definition of f, so $y = f(x) \in \rho^{-1}_j f_j \pi_{i_j}(U)$. Now suppose that $y \in \rho^{-1}_j f_j \pi_{i_j}(U)$. We will construct a point $x \in U$ such that $f(x) = y$. For each k, let $y_k = \rho_k(y)$. Now for each $k > j$, we claim the following two statements are true:

1) $\pi^{-1}_{i_k} f^{-1}_k(y_k)$ is closed in K_π.

This is easy to see, since the set in question is the continuous preimage of a singleton, which is closed in I_k.

2) If $k > n$, then $\pi^{-1}_{i_k} f^{-1}_k(y_k) \subset \pi^{-1}_{i_n} f^{-1}_n(y_n)$.

To see this, suppose that $p \in \pi^{-1}_{i_k} f^{-1}_k(y_k)$. Then $f_k \pi_{i_k}(p) = y_k$. Next, $f_n \pi^{i_n}_{i_k} \pi_{i_k}(p) = \rho^n_k f_k \pi_{i_k}(p)$. But this yields $f_n \pi_{i_n}(p) = \rho^n_k(y_k) = y_n$, so $p \in \pi^{-1}_{i_n} f^{-1}_n(y_n)$.

Since 1) and 2) hold, we know that the set $\bigcap_{k>j} \pi_{i_k}^{-1} f_k^{-1}(y_k)$ is nonempty and contains some point x. For each $k > j$, $\rho_k f(x) = f_k \pi_{ik}(x)$. Since $x \in \pi_{i_k}^{-1} f_k^{-1}(y_k)$, we know that $f_k \pi_{i_k}(x) = y_k$. Also, for each $k < j$, $\rho_k f(x) = f_k \pi_{i_k}(x) = \rho_j^k f_j \pi_{i_j}(x) = \rho_j^k(y_j) = y_k$, so $f(x) = y$. ∎

K_π is said to be an **even** Knaster continuum if $occ_\pi(2) = \infty$, otherwise it is an **odd** Knaster continuum.

In order to simplify matters we will require that, when choosing a representative K_ρ of an odd Knaster continuum, the sequence ρ contains no 2's at all, i.e., $occ_\pi(2) = 0$.

3.2 Lemma *If a sequence i_k of indices and maps $f_k : I_{i_k} \to I_k$ induces an open map $f \colon K_\pi \to K_\rho$, then $f_k \in \mathcal{O}^+$ for all k or $f_k \in \alpha\mathcal{O}^+$ for all k.*

Proof: If K_ρ is an even Knaster continuum, it follows from part i) of Lemma 2.2 that all the maps f_k are order preserving. For if f_k is order reversing for some k, then choosing $l > k$ so large that ρ_k^l has even degree, we have $\rho_k^l f_l = f_k \pi_{i_k}^{i_l}$. But $\rho_k^l f_l(0) = \rho_k^l(1) = 0$ while $f_k \pi_{i_k}^{i_l}(0) = f_k(0) = 1$, a contradiction.

If K_ρ is an odd Knaster continuum (with no 2's in ρ), then it follows from part ii) of Lemma 2.2 that $f_k(0) = f_1(0)$ for all k, so all the maps f_k are order preserving or all maps are order reversing. ∎

3.3 Lemma *If a sequence i_k of indices and maps $f_k : I_{i_k} \to I_k$ induces an open map $f \colon K_\pi \to K_\rho$, then the map f is completely determined by any map in the defining sequence.*

Proof: Fix a map $f_n : I_{i_n} \to I_n$ in the defining sequence for f and suppose that $g : K_\pi \to K_\rho$ is an induced open map with a defining sequence j_k of indices and maps $g_k : I_{j_k} \to I_k$ in which $j_n = i_n$ and $g_n = f_n$. It is required to show that $g = f$. Let $x = (x_1, x_2, \cdots) \in K_\pi$. Then $f(x) = (y_1, y_2, \cdots) \in K_\rho$ and $g(x) = (z_1, z_2, \cdots) \in K_\rho$ where we know that $y_n = f_n(x_{i_n}) = g_n(x_{i_n}) = z_n$. Hence $y_k = z_k$ for $k = 1, \cdots, n$. Let $k > n$, and assume without loss of generality that $j_k \geq i_k$. Then we have

$$f_n \pi_{i_n}^{j_k} = g_n \pi_{i_n}^{j_k} = \rho_n^k g_k$$

since $f_n = g_n$ and g is an induced map. But also we have

$$f_n \pi_{i_n}^{j_k} = f_n \pi_{i_n}^{i_k} \pi_{i_k}^{j_k} = \rho_n^k f_k \pi_{i_k}^{j_k}$$

since f is an induced map. Hence $\rho_n^k g_k = \rho_n^k f_k \pi_{i_k}^{j_k}$. Now by Lemma 3.2, all the maps in the defining sequence for f are order preserving or all the maps are order reversing. The same is true for g, and since $g_n = f_n$ we can apply parts 1 and 2 of Lemma 2.6 to cancel ρ_n^k on the left and get $g_k = f_k \pi_{i_k}^{j_k}$. Hence

$$y_k = f_k(x_{i_k}) = f_k \pi_{i_k}^{j_k}(x_{j_k}) = g_k(x_{j_k}) = z_k.$$

This shows that $f(x) = g(x)$ for all $x \in K_\pi$ and completes the proof that $f = g$. ∎

Given an $f \in \mathcal{O}^+$, and an integer k, let $(f)_1^k$ be the map f considered as a map from I_k to I_1. Now $(f)_1^k$ may or may not be the first term in a defining sequence of maps for some induced open map from K_π to K_ρ. If it is, we use the symbol $\overline{(f)_1^k}(\pi, \rho)$ to stand for the induced map. If it is clear from the context, we will drop the reference to π and ρ. Also, $\overline{f}$ is used as an abbreviation of $\overline{(f)_1^1}(\pi, \pi)$.

Note: It will shorten some statements if we agree that $\pi_1^1 = w_1$, the identity map on I.

3.4 Lemma *Let K_π and K_ρ be Knaster continua.*

1) If $f_k\colon I_{i_k} \to I_k$ is a defining sequence for an open induced map $f \in \mathcal{OI}_\rho^\pi$, then for each $n \geq 1$, $f = \overline{(\rho_1^n f_n)_1^{i_n}}(\pi, \rho)$.

2) For each $f \in \mathcal{O}^+$ and each integer $n \geq 1$, $\overline{(\pi_1^n f)_1^n} = \overline{(\pi_1^n f)_1^n}(\pi, \pi)$ exists. In particular, $\overline{(\pi_1^n)_1^n}$ is the identity map on K_π. In addition, if $g \in \mathcal{O}^+$, then $\overline{(\pi_1^n g)_1^n}\ \overline{(\pi_1^n f)_1^n} = \overline{(\pi_1^n g f)_1^n}$. Further, if f is a homeomorphism then $\overline{(\pi_1^n f)_1^n}$ is a homeomorphism.

3) If π is an odd sequence with $occ_\pi(2) = 0$, then $\overline{\alpha}$ exists. If π is even, then $\overline{\alpha}$ does not exist.

Proof: 1) This identity is established by applying both maps to an arbitrary point $x = (x_1, x_2, \cdots) \in K_\pi$.

$$f(x) = (f_1(x_{i_1}), f_2(x_{i_2}), \cdots,) = (\rho_1^n f_n(x_{i_n}), \cdots) = \overline{(\rho_1^n f_n)_1^{i_n}}(x).$$

2) Let $p_1 = f$ and apply 2.4 repeatedly to construct a sequence of open maps $p_k\colon I_{n+k-1} \to I_{n+k-1}$ so that $\pi_{n+k-1}^{n+k} p_{k+1} = p_k \pi_{n+k-1}^{n+k}$, for $k \geq 1$. Define

$$f_k = \pi_k^{n+k-1} p_k\colon I_{n+k-1} \to I_k \text{ for each } k.$$

This sequence induces a map $F\colon K_\pi \to K_\pi$ which is open because all its coordinate maps are open (3.1). Further, by part 1), $F = \overline{(\pi_1^n f)_1^n}$ and so $\overline{(\pi_1^n f)_1^n}$ exists. To see that $\overline{(\pi_1^n)_1^n}$ is the identity map on K_π, apply the map to a point $(x_1, x_2, \cdots,) \in K_\pi$:

$$\overline{(\pi_1^n)_1^n}(x_1, x_2, \cdots) = (\pi_1^n(x_n), \cdots) = (x_1, \cdots).$$

If $g \in \mathcal{O}^+$, then after constructing the defining sequences $g_k = \pi_k^{n+k-1} q_k$ and $(gf)_k = \pi_k^{n+k-1} s_k$ (with the maps q_k and s_k defined analogously to the maps p_k) for the maps $\overline{(\pi_1^n g)_1^n}$ and $\overline{(\pi_1^n g f)_1^n}$, note that

$$\begin{aligned}
\overline{(\pi_1^n g)_1^n}\ \overline{(\pi_1^n f)_1^n}(x_1, \cdots, x_{2n-1}, \cdots) &= \overline{(\pi_1^n g)_1^n}(\pi_1^n f(x_n), \cdots, \pi_n^{2n-1} p_{2n-1}(x_{2n-1}), \cdots) \\
&= (\pi_1^n g \pi_n^{2n-1} p_{2n-1}(x_{2n-1}), \cdots) = (\pi_1^n g f \pi_n^{2n-1}(x_{2n-1}), \cdots) \\
&= (\pi_1^n g f(x_n), \cdots) = \overline{(\pi_1^n g f)_1^n}(x_1, \cdots)
\end{aligned}$$

Finally, if f is a homeomorphism of I, then by what has just been shown,

$$\overline{(\pi_1^n f)_1^n}\ \overline{(\pi_1^n f^{-1})_1^n} = \overline{(\pi_1^n f^{-1})_1^n}\ \overline{(\pi_1^n f)_1^n} = \overline{(\pi_1^n)_1^n},$$

and $\overline{(\pi_1^n)_1^n}$ is the identity map on K_π.

3) In case π has no 2's, the sequence $f_k = \alpha$ induces an open map $\overline{\alpha}$ on K_π using Lemma 2.2. If π is an even sequence, then no order reversing map can induce an open map on K_π, again by Lemma 2.2. ∎

Let n be a positive integer. An induced map $g \in \mathcal{OI}_\pi$ is said to be **vertically induced with order at most n** provided $g = \overline{(\pi_1^n f)_1^n}$ for some $f \in \mathcal{O}$. The **order** of a vertically induced map is the smallest n for which it is vertically induced of order at most n. The next theorem shows that there are lots of isomorphisms of $\mathcal{O}^+$ into $\mathcal{O}_\pi$.

3.5 Theorem *For each positive integer n, define $F_n : \mathcal{O}^+ \to \mathcal{O}_\pi$ by $F_n(f) = \overline{(\pi_1^n f)_1^n}$. Then F_n is an isomorphism of $\mathcal{O}^+$ with the set of vertically induced open maps of order at most n. The set of images $F_n(\mathcal{O}^+)$ is an increasing tower; that is, $F_n(\mathcal{O}^+) \subset F_{n+1}(\mathcal{O}^+)$. Finally, if $occ_\pi(2) = 0$, then F_n extends to all of $\mathcal{O}$*

Proof: That F_n is a well-defined homomorphism follows from parts 1) and 2) of Lemma 3.4. To see that F_n is 1-1, suppose $F_n(f) = F_n(g)$. Then the first terms of the defining sequences for these maps are equal, i.e., $\pi_1^n f = \pi_1^n g$. But $\mathcal{O}^+$ is (left) cancellative, so $f = g$. To see that $F_n(\mathcal{O}^+) \subset F_{n+1}(\mathcal{O}^+)$, note that

$$F_n(f) = \overline{(\pi_1^n f)_1^n} = \overline{(\pi_1^n f \pi_n^{n+1})_1^{n+1}} = \overline{(\pi_1^n \pi_n^{n+1} p_2)_1^{n+1}} = \overline{(\pi_1^{n+1} p_2)_1^{n+1}} = F_{n+1}(p_2)$$

Finally, assume π is a sequence of odd primes. Then by Lemma 2.2, α commutes with all the bonding maps of K_π, and hence induces an open map $\overline{\alpha}\colon K_\pi \to K_\pi$. By the structure theorem for $\mathcal{O}$, Theorem 2.4, each open map $f \in \mathcal{O}$ which is not order preserving looks like αg where $g = \alpha f \in \mathcal{O}^+$, and hence maps to $\overline{\alpha g}$. ∎

Let $\mathcal{OV}_\pi$ be the union of the tower of subsemigroups $F_n(\mathcal{O}^+)$ ($F_n(\mathcal{O})$ if π is odd with no 2's). Then it follows from Theorem 3.5 that $\mathcal{OV}_\pi$ is a subsemigroup of $\mathcal{OI}_\pi$, which we refer to as the **semigroup of open vertically induced maps** of K_π. Similarly, let $\mathcal{HV}_\pi$ be the union of the increasing tower of groups $F_n(\mathcal{H}^+)$. By part 2 of 3.4, the maps in $\mathcal{HV}_\pi$ are homeomorphisms of K_π. Using 3.5, we can see that $\mathcal{HV}_\pi$ is a subgroup of the group of units of $\mathcal{O}_\pi$. We refer to it as the **group of vertically induced homeomorphisms** of K_π.

Note that for any m, n, $F_n(w_m) = F_1(w_m) = \overline{w_m}$, so the image of the standard maps $\mathcal{W}$ remains the same under the isomorphisms F_n. We denote this common image by $\mathcal{W}_\pi$ and refer to it as the **semigroup of standard induced maps** on K_π.

The next lemma gives a factorization of an arbitrary open induced map from K_π to K_ρ.

3.6 Lemma *Let $g \in \mathcal{OI}_\rho^\pi$. Then g can be factored into $\overline{\alpha}^i q v$ where $i \in \{0, 1\}$, $q = \overline{(w_m)_1^n}(\pi, \rho)$, and $v \in \mathcal{HV}_\pi$.*

Proof: Let i_k and $g_k\colon I_{i_k} \to I_k$ be a sequence of indices and maps inducing g. First, by Theorem 2.4, factor $g_k = \alpha^{j_k} w_{m_k} h_k$, where h_k is an order preserving homeomorphism. By Lemma 3.2, we know that $j_k = 0$ for all k or $j_k = 1$ for all k. Denote this common value by

j. Let $v = \overline{(\pi_1^{i_1} h_1)_1^{i_1}}(\pi,\pi) = F_{i_1}(h_1)$. This vertically induced homeomorphism exists from Lemma 3.4. Next, note that for each k,

$$g_k \pi_{i_k}^{i_{k+1}} = \rho_k^{k+1} g_{k+1}.$$

Substituting in the factorizations, we have

$$\alpha^j w_{m_k} h_k \pi_{i_k}^{i_{k+1}} = \rho_k^{k+1} \alpha^j w_{m_{k+1}} h_{k+1}.$$

If $j = 0$, we can erase the α^j on both sides of the equation. If $j = 1$, then ρ_k^{k+1} is odd and $\alpha = \alpha^j$ commutes with it by 2.2, so we can multiply both sides of the equation by α and erase it. In either case, we have

$$w_{m_k} h_k \pi_{i_k}^{i_{k+1}} = \rho_k^{k+1} w_{m_{k+1}} h_{k+1}.$$

Now $h_k \pi_{i_k}^{i_{k+1}} = \pi_{i_k}^{i_{k+1}} h_{k+1}$, and so

$$w_{m_k} \pi_{i_k}^{i_{k+1}} h_{k+1} = \rho_k^{k+1} w_{m_{k+1}} h_{k+1}.$$

Now multiply on the right by $(h_{k+1})^{-1}$ and obtain

$$w_{m_k} \pi_{i_k}^{i_{k+1}} = \rho_k^{k+1} w_{m_{k+1}}.$$

We have shown that $q = \overline{(w_{m_1})_1^{i_1}}(\pi,\rho)$ exists. If $j = 1$, let $\overline{\alpha}^j = \overline{\alpha}(\rho,\rho)$, which we know exists because ρ is odd with no 2's. If $j = 0$, let $\overline{\alpha}^j$ be $\overline{w_1}(\rho,\rho)$. In either case, we can calculate that

$$\overline{\alpha}^j \overline{(w_{m_1})_1^{i_1}}(\pi,\rho) \overline{(\pi_1^{i_1} h_1)_1^{i_1}}(\pi,\pi) = g. \blacksquare$$

One consequence of 3.6 is that there is an open induced map from K_π to K_ρ (if and) only if there is one of the form $\overline{(w_m)_1^k}(\pi,\rho)$ for some m and k.

For each positive integer n and Knaster continua K_π and K_ρ, define a function

$$d_n(\pi,\rho) : \mathbb{P} \to \omega$$

which we will call the **n deficit of** π **over** ρ, by $d_n(\pi,\rho)(p) = \max\{0, occ_\rho(p) - occ_{(\pi_i)_{i=n}^\infty}(p)\}$. We will say that $d_n(\pi,\rho)$ is **trivial** if it never takes ∞ as a value and all but finitely many of its values are 0.

The next lemma tells when $\overline{(w_m)_1^n}(\pi,\rho)$ exists and gives a factorization of it which will prove useful.

3.7 Lemma *The map $\overline{(w_m)_1^n}(\pi,\rho)$ exists if and only if $d_n(\pi,\rho)$ is trivial and $m = dt$ for some integer t, where $d = \prod_{p\in\mathbb{P}} p^{d_n(\pi,\rho)(p)}$. In this case, $\overline{(w_m)_1^n}(\pi,\rho) = \overline{(w_d)_1^n}(\pi,\rho)\overline{(w_t)_1^1}(\pi,\pi)$.*

Proof: First suppose that $\overline{(w_m)_1^n}(\pi,\rho)$ exists.

Let $f_k = w_{m_k} : I_{n_k} \to I_k$ be a defining sequence for $\overline{(w_m)_1^n}(\pi,\rho)$. Suppose d does not divide m. Then there is a prime p such that the highest power p^j that divides d does not divide m. Choose k so large that if $\rho_i^{i+1} = w_p$ then $i < k$ and if $\pi_i^{i+1} = w_p$, then $i < n$. Let p^l and p^s be the highest powers of p dividing $deg(\rho_1^k)$ and $deg(\pi_n^{n_k})$ respectively. Then by the definition of d, p does not divide f_k, and so $p^j p^s = p^l$. But $m\ deg(\pi_n^{n_k}) = deg(\rho_1^k) f_k$. It follows that p^j must divide m, a contradiction.

Now suppose that the condition holds. We will show that $\overline{(w_d)_1^n}$ exists. Let $f_1 = w_d : I_n \to I_1$, and suppose $f_k : I_{n_k} \to I_k$ has been defined so that $f_{k-1}\ \pi_{n_{k-1}}^{n_k} = \rho_{k-1}^k\ f_k$. Let p be the prime such that $\rho_k^{k+1} = w_p$. Let p^j, p^s and p^l be the highest powers of p dividing m, $deg(\pi_n^{n_k})$, and $deg(\rho_1^k)$ respectively. If $\pi_i^{i+1} \neq w_p$ for all $i \geq n_k$, then $l < j + s$. Hence p divides $deg(f_k)$ and so we can choose $n_{k+1} = n_k + 1$ and define $f_{k+1} = w_r$ where $r = \dfrac{deg(f_k)}{p}\ deg(\pi_{n_k}^{n_k+1})$. Otherwise, choose $i > n_k$ so that $\pi_i^{i+1} = w_p$, and define $n_{k+1} = i + 1$ and $f_{n_{k+1}} = w_r$, where $r = \dfrac{deg(f_k)\ deg(\pi_1^{n_k})}{p}$. Thus $\overline{(w_d)_1^n}(\pi, \rho)$ exists. Now $\overline{(w_{dt})_1^n}(\pi, \rho) = \overline{(w_d)_1^n}(\pi, \rho)\overline{(w_t)_1^1}(\pi, \pi)$ exists. ■

Now if $\pi = \rho$, the result of Lemma 3.7 can be sharpened. As we shall see, the map $\overline{(w_d)_1^n}(\pi, \pi)$ can be factored nicely. First we need some invertibility lemmas.

3.8 Lemma *Suppose that p and q are distinct prime numbers and p is odd. Then w_p permutes each of $w_q^{-1}(0)$ and $w_q^{-1}(1)$.*

Proof: First, when $n = 2$, w_p fixes each of $w_n^{-1}(0)$ and $w_n^{-1}(1)$, so the result is trivially true.

Now suppose that n is odd. Note that for each $x \in w_n^{-1}(0)$, that $x = \frac{2k}{n}$ for some $0 \leq k \leq \frac{n-1}{n}$, and that either $w_p(x) = -i + \frac{p \cdot 2k}{n} = \frac{-ni + 2pk}{n}$ for some $i \in 2N$ or $w_p(x) = i + 1 - p \cdot \frac{2k}{n} = \frac{n(i+1) - 2pk}{n}$ for some $i + 1 \in 2N$. In either case, there is an integer r such that $w_p(x) = \frac{2(r - pk)}{n} \in I \cup w_n^{-1}(0)$. Similarly, if $x \in w_n^{-1}(1)$ it can be shown that for some integer r, that $w_p(x) = \frac{2(r-pk)+1}{n} \in I \cup w_n^{-1}(1)$. So $w_p(w_n^{-1}(0)) \subset w_n^{-1}(0)$ and $w_p(w_n^{-1}(1)) \subset w_n^{-1}(1)$.

We now show that w_p is one to one on $w_n^{-1}(0) \cup w_n^{-1}(1)$. Suppose that for some $0 \leq a, b, \leq n$, there are points $\frac{a}{n}$ and $\frac{b}{n}$ such that $w_p(\frac{a}{n}) = w_p(\frac{b}{n})$. By the definition of w_p, there are three cases to consider:

1. w_p has positive slope at both $\frac{a}{n}$ and $\frac{b}{n}$. Then there are natural numbers i and k so that $-i + \frac{pa}{n} = -b + \frac{pb}{n}$. This means that $\frac{pa}{n} - \frac{pb}{n} = \frac{p(a-b)}{n}$ is an integer. Since n and p are relatively prime, we know that n divides $a - b$. Now, since $0 \leq a, b, \leq n$, we know that either $a = b$ or that $a \in \{0, n\}$ and $b = n - a$. If $a = 0$ and $b = n$, then $w_p(a/n) = w_p(0) = 0 \neq w_p(1) = w_p(b/n)$. This means that it must be the case that $a = b$.

2. w_p has negative slope at both $\frac{a}{n}$ and $\frac{b}{n}$. This case is essentially the same as case 1.

3. w_p has positive slope at one of $\{\frac{a}{n}, \frac{b}{n}\}$ and negative slope at the other. We will assume the notation is chosen so that w_p has positive slope at $\frac{b}{n}$. Then there are natural

numbers i and j so that $i+1-\frac{pa}{n} = -k+\frac{pb}{n}$. In this case, $\frac{pa}{n}+\frac{pb}{n} = \frac{p(a+b)}{n}$ is an integer. Since p and n are distinct primes, we know that n divides $a+b$. Now, since $0 \leq a, b, \leq n$, we have one of the following cases to consider:

(a) $a+b=0$. Then $a=b=0$, so $a/n = 0 = b/n$.

(b) $a+b=2n$. Then $a=b=n$, so $a/n = 1 = b/n$.

(c) $a+b=n$. Then $0<a<n$ and $b=n-a$. This means that $\frac{b}{n} = 1-\frac{a}{n}$. Since p is odd, and therefore the graph of w_p is symmetric about the point $(1/2, 1/2)$, it follows that w_p has the same slope at a/n as it does at $1-a/n = b/n$. So this case is impossible.

Now, since w_p takes each of $w_n^{-1}(1)$ and $w_n^{-1}(0)$ into itself, and since w_p is one to one on $w_n^{-1}(0) \cup w_n^{-1}(1)$, we know that w_p permutes each of these sets.

Note, in particular, that if p and n are distinct primes and p is odd, then w_p permutes $w_n^{-1}(0)$. ■

3.9 Lemma *If n is an odd prime, then w_2 maps each of $w_n^{-1}(0)$ and $w_n^{-1}(1)$ one to one onto $w_n^{-1}(0)$. In particular, w_2 permutes $w_n^{-1}(0)$.*

Proof: Note that $1/2 \notin w_n^{-1}(0) \cup w_n^{-1}(1)$, because n is odd. We first show that $w_2(w_n^{-1}(0)) \subset w_n^{-1}(0)$. Note that $x \in w_n^{-1}(0)$ if and only if for some $0 \leq k \leq \frac{n-1}{2}$, that $x = \frac{2k}{n}$. If $x < 1/2$, then $w_2(x) = \frac{2(2k)}{n} \in w_n^{-1}(0)$, and if $x > 1/2$, then

$$w_2(x) = 2 - \frac{2(2k)}{n} = \frac{2(n-2k)}{n} \in w_n^{-1}(0).$$

We next show that w_2 is one to one on $w_n^{-1}(0)$. To see this, first note that w_2 is one to one on each of $w_n^{-1}(0) \cup [0, 1/2]$ and $w_n^{-1}(0) \cup [1/2, 1]$. Now if $x \in w_n^{-1}(0) \cup [0, 1/2]$, then $x = \frac{2k}{n}$ for some k and $w_2(x) = \frac{4k}{n}$. Since n is odd, we know that the numerator of this expression is an even multiple of 2. Now, if $x \in [1/2, 1] \cup w_n^{-1}(0)$, we have that $w_2(x) = 2 - \frac{2k}{n} = \frac{2n-4k}{n} = \frac{2(n-2k)}{n}$ for some natural number k. Since n is odd, we see that the numerator of this expression is an odd multiple of 2. Therefore, $w_2(w_n^{-1}(0) \cup [0, 1/2]) \cup w_2(w_n^{-1}(0) \cup [1/2, 1]) = \emptyset$ and w_n permutes $w_n^{-1}(0)$.

Finally, since for each $x \in I$, $w_2(x) = w_2(1-x)$, and since the function $\alpha(x) = 1-x$ is a bijection from $w_n^{-1}(1)$ onto $w_n^{-1}(0)$, w_2 maps $w_n^{-1}(1)$ one to one onto $w_n^{-1}(0)$. ■

A standard map w_n on I is not invertible in $\mathcal{O}$. However, its image $\overline{w_n} \in \mathcal{O}_\pi$ will be invertible when the prime factors of n occur infinitely often in π, i.e., $occ_\pi(p) = \infty$ for each prime divisor p of n.

3.10 Invertibility Theorem *The standard induced map $\overline{w_n}$ is invertible in $\mathcal{O}_\pi$ if and only if for each prime factor p of n, $occ_\pi(p) = \infty$. Furthermore, if p is a prime such that $occ_\pi(p) = \infty$, then $\overline{w_p}^{\;-1} = \overline{(\pi_1^{k-1})_1^k}$, where k is chosen so that $\pi_{k-1}^k = w_p$.*

Proof: First, suppose that the condition fails. Without loss of generality, we can assume that some prime factor p of n does not occur in π at all. We show that that $\overline{w_p}$ is not 1-1. It is clear that $\overline{w_p}((0,0,0...)) = (0,0,0,...)$. By lemmas 3.8 and 3.9, for each π_i^{i+1} of K_π, π_i^{i+1} permutes $w_p^{-1}(0)$. Thus, there is at least one point $x = (\frac{2}{p}, x_2, x_3, ...) \neq (0,0,0,...) \in K_\pi$ for which $x_i \in w_p^{-1}(0)$ for each i, and so $\overline{w_p}(x) = (0,0,0,...)$. Hence $\overline{w_p}$ is not 1-1 and so is not invertible. But this implies that $\overline{w_n}$ is not invertible, since $\overline{w_p}$ is a factor of it. This completes the proof of the only if part.

Now suppose that $\overline{w_n}$ is invertible. It is enough to assume that n is a prime; since if p and q are primes with $\overline{w_p}$ and $\overline{w_q}$ invertible, then $\overline{w_p}\ \overline{w_q} = \overline{w_{pq}}$ is invertible. When n is prime, we know that it occurs infinitely often in π, so there is an increasing sequence of integers, $1 < k_1 < k_2 < ...$ for which $\pi_{k_i-1}^{k_i} = w_n$. For each i, define $g_i : I_{k_i} \to I_i$ by $g_i = \pi_i^{k_i-1}$. Note that for each i,

$$g_i\pi_{ki}^{k_{i+1}} = \pi_i^{k_i-1}\pi_{k_i}^{k_{i+1}} = \pi_i^{k_i-1}\pi_{k_i}^{k_{i+1}-1}w_n = \pi_i^{k_i-1}w_n\pi_{k_i}^{k_{i+1}-1} =$$

$$\pi_i^{k_i-1}\pi_{k_i-1}^{k_i}\pi_{k_i}^{k_{i+1}-1} = \pi_i^{k_{i+1}-1} = \pi_i^{i+1}\pi_{i+1}^{k_{i+1}-1} = \pi_i^{i+1}g_{i+1},$$

so the sequence of maps g_i induces a map $g\colon K_\pi \to K_\pi$. Finally, note that for each i,

$$\pi_i g\overline{w_n}(x) = g_i\pi_{k_i}\overline{w_n}(x) = g_i w_n\pi_{k_i}(x) = \pi_i^{k_i-1}\pi_{k_i-1}^{k_i}\pi_{k_i}(x) = \pi_i^{k_i}\pi_{k_i}(x) = \pi_i(x),$$

and hence $g\overline{w_n} = \overline{w_1}$, the identity map on K_π. So $\overline{w_n}^{-1}$ exists and equals $g = \overline{(g_1)_1^{k_1}} = \overline{(\pi_1^{k_1-1})_1^{k_1}}$. ■

In particular, note that when π is the constant sequence n, then $\overline{w_n}^{-1}$ is the shift map, $s\colon K_\pi \to K_\pi$, defined by $s(x_1, x_2, x_3, ...) = (x_2, x_3, ...)$.

We can now state an existence and factorization theorem for maps $\overline{(w_n)_1^k}(\pi,\pi)$.

3.11 Theorem *Write $\pi_1^k = w_s w_f$, where for each prime factor p of s, $occ_\pi(p) < \infty$ and for each prime factor p of f, $occ_\pi(p) = \infty$. Then $\overline{(w_n)_1^k}(\pi,\pi)$ exists if and only if $n = st$ for some t. In that case, $\overline{(w_n)_1^k}(\pi,\pi) = \overline{w_t}\ \overline{w_f}^{\ -1}$.*

Proof: The first statement follows from Lemma 3.7 upon noting that the d in that lemma is the s of this corollary. The second statement follows from the factorization given in Lemma 3.7 and the Invertibility Theorem 3.10. ■

Theorem 3.10 also enables us to answer affirmatively the question raised in the previous section about the embeddability of $\mathcal{O}^+$ into a group. Let γ denote the sequence $2, 3, 2, 3, 5, 2, 3, 5, 7, \cdots$ of primes in which each prime occurs infinitely often.

3.12 Corollary *The induced open maps of K_γ form a group. Hence $\mathcal{O}^+$ is embeddable into the group of units of K_γ.*

Proof: Each prime occurs infinitely often in γ, and so for each positive integer n, 3.10 says that $\overline{w_n}$ is invertible in $\mathcal{O}_\gamma$, hence the isomorphism F_1 takes $\mathcal{O}^+$ into the group of units of $\mathcal{O}_\gamma$. ■

In [6], Debski defines the degree of an arbitrary open map between Knaster continua. For the moment, we will define the degree of an induced map in a simpler fashion. Later, in the next section, we show that the two definitions agree on the induced open maps.

Suppose K_ρ and K_π are Knaster continua with $K_\rho \le K_\pi$. For any map $f \in \mathcal{OI}_\rho^\pi$, define the **degree** of f, by $deg(f) = \dfrac{deg(f_1)}{deg(\pi_1^{i_1})}$, where $f_1 \colon I_{i_1} \to I_1$ is the first coordinate map of f and π_1^1 is by decree w_1.

3.13 Theorem *1) If $f \in \mathcal{OI}_\rho^\pi$ and $g \in \mathcal{OI}_\delta^\rho$, then $deg(gf) = deg(g)\ deg(f)$.*

2) If $\pi = \rho = \delta$, then $deg \colon \mathcal{OI}_\pi \to Q$ is a homomorphism into the group Q^+ of positive rational numbers under multiplication.

3) The open induced maps with degree 1 consist precisely of the vertically induced homeomorphisms $\mathcal{HV}_\pi$, and the open induced maps of positive integer degree consist precisely of the open vertically induced maps $\mathcal{OV}_\pi$.

4) The image $deg(\mathcal{OI}_\pi)$ is the subsemigroup Q_π of Q^+ consisting of all positive rationals $\frac{n}{m}$ such that for each prime divisor p of m, $occ_\pi(p) = \infty$.

Proof: 1) Let $f_k \colon I_{i_k} \to I_k$ and $g_l \colon I_{j_l} \to I_l$ be defining sequences for f and g. Now

$$gf = \overline{(g_1)_1^{j_1}}\ \overline{(f_1)_1^{i_1}} = \overline{(g_1)_1^{j_1}}\ \overline{(\rho_1^{j_1} f_{j_1})_1^{i_{j_1}}} = \overline{(g_1\ f_{j_1})_1^{i_{j_1}}}.$$

Hence, the degree of gf is

$$deg(gf) = \frac{deg(g_1\ f_{j_1})}{deg(\pi_1^{i_{j_1}})} = \frac{deg(g_1)}{deg(\rho_1^{j_1})} \frac{deg(\rho_1^{j_1}) deg(f_{j_1})}{deg(\pi_1^{i_{j_1}})} = deg(g)\ deg(f).$$

2) This follows immediately from 1).

3) Let $f \in \mathcal{HV}_\pi$. Then $f = \overline{(\pi_1^{i_1} h)_1^{i_1}}$, where $h \in \mathcal{H}$. Then $deg(f) = \dfrac{deg(\pi_1^{i_1} h)}{deg(\pi_1^{i_1})} = deg(h) = 1$. Conversely suppose $f \in \mathcal{OV}_\pi$ has degree 1. Let $f_1 = \alpha^j w_m g \colon I_{i_1} \to I_1$ be the first coordinate map of f, where g is a homeomorphism of I. Then $1 = deg(f) = \dfrac{deg(f_1)}{deg(\pi_1^{i_1})} = \dfrac{m}{deg(\pi_1^{i_1})}$, hence $w_m = \pi_1^{i_1}$, and $f \in \mathcal{HV}_\pi$

4) Let $f \in \mathcal{OV}_\pi$, then by 3.7, $f = \alpha^j qv$ where $q = \overline{(w_m)_1^n}(\pi, \pi)$, and $v \in \mathcal{HV}_\pi$. Hence by the results of the above paragraphs, $deg(f) = deg(q)$. But now, by 3.7, q factors into $\overline{(w_d)_1^m}\,\overline{(w_t)_1^1}$, where $d = \prod_{p \in \mathbb{P}} p^{d(m,\pi,\pi)(p)}$ and $m = dt$. Hence $deg(q) = deg(\overline{(w_d)_1^m}) deg(\overline{(w_t)_1^1}) = \dfrac{d}{deg(\pi_1^m)} t$. Let $deg(\pi_1^m) = M$. Now, by 3.11, $d = s$ divides M and we can write $\pi_1^m = w_M = w_d w_k$, where for each prime factor p of k, $occ_\pi(p) = \infty$ and for each prime factor p of d, $occ_\pi(p) < \infty$. Hence $deg(q) = \dfrac{d}{deg(\pi_1^m)} t = \dfrac{d}{dk} t = \dfrac{t}{k} \in Q_\pi$. All that is left is to show that

each $\frac{t}{k} \in Q_\pi$ is the degree of some open induced map. This follows from the easily established facts that 1) Q_π is generated by the primes and the reciprocals of the primes p which occur infinitely often in π, and 2) if p is a prime, then $deg(\overline{w_p}) = p$ and if $occ_\pi(p) = \infty$, then $deg(\overline{w_p}^{\,-1}) = \frac{1}{p}$. ■

Let $\overline{\mathcal{W}}^*_\pi$ denote the subsemigroup of $\mathcal{OI}_\pi$ generated by the induced standard maps $\overline{w_n}$ together with $\overline{w_p}^{\,-1}$ where $occ_\pi(p) = \infty$. The proof of the following theorem is immediate.

3.14 Theorem *The function deg takes* $\overline{\mathcal{W}}^*_\pi$ *isomorphically onto* Q_π. *Hence,*

1) $\overline{\mathcal{W}}^*_\pi$ *is commutative.*

2) Each element f *of* $\overline{\mathcal{W}}^*_\pi$ *can factored uniquely as* $f = \overline{w_m}\ \overline{w_n}^{\,-1}$ *where* m *and* n *are relatively prime. Further, if* $f = \overline{w_m}\ \overline{w_n}^{\,-1}$ *and* $g = \overline{w_s}\ \overline{w_t}^{\,-1}$ *are in* $\overline{\mathcal{W}}^*_\pi$ *then* $f\ g = (\overline{w_m}\ \overline{w_n}^{\,-1})(\overline{w_s}\ \overline{w_t}^{\,-1}) = \overline{w_{ms}}\ \overline{w_{nt}}^{\,-1}$.

3) If $\overline{w_n}$ *is invertible in* $\mathcal{O}_\pi$ *and* $f = \overline{w_m}\ \overline{w_n}^{\,-1}$, *then* $deg(f) = \frac{m}{n}$.

We now want to introduce some notation. Given a rational number $\frac{m}{n} \in Q_\pi$ with $gcd(m,n) = 1$, let $w_{\frac{m}{n}}$ denote $\overline{w_m}\ \overline{w_n}^{\,-1}$. Further if $v = \overline{(\pi_1^{i_1} h)_1^{i_1}}$ is a vertically induced homeomorphism, then mv is defined to be the vertically induced homeomorphism $\overline{(\pi_1^{i_1}(mh))_1^{i_1}}$, where mh is the multiple of h defined above 2.4. We also want to define $\frac{1}{n}v$. First, we define $\frac{1}{p}v$, where $occ_\pi(p) = \infty$, as follows: Choose $k > 1$ so large that $\pi_{i_k}^{i_{k+1}} = w_p$. Then $v = \overline{(\pi_1^{i_{k+1}} h_{k+1})_1^{i_{k+1}}}$. By the definition above 2.4, $h_k = \frac{1}{p}h_{k+1}$, and we define $\frac{1}{p}v = \overline{(\pi_1^{i_{k+1}} h_k)_1^{i_{k+1}}}$. Now $\frac{1}{n}v$ is defined by induction on the sum of the exponents of the prime factors of n.

Figure 2: $v\overline{w_p}^{\,-1} = \overline{w_p}^{\,-1}(\frac{1}{p}v)$

$$
\begin{tikzcd}
I_1 & \cdots & I_{k-1} & I_k \arrow[l, "w_p"'] \arrow[dlll, "\pi_1^{k-1}"'] \arrow[d, "\frac{1}{p}h"'] & \cdots & I_{n-1} & I_n \arrow[l, "w_p"'] \arrow[dlll, "\pi_k^{n-1}"'] \arrow[d, "g"'] \\
I_1 & \cdots & I_{k-1} \arrow[d, "\frac{1}{p}h"'] & I_k \arrow[l, "w_p"'] \arrow[d, "h"'] & \cdots & I_{n-1} & I_n \arrow[l, "w_p"'] \arrow[dlll, "\pi_k^{n-1}"'] \\
I_1 & \cdots & I_{k-1} & I_k \arrow[l, "w_p"'] & & &
\end{tikzcd}
$$

3.15 Lemma *Let* $v \in \mathcal{HV}_\pi$ *and let* m *and* p *be integers, where* p *is a prime with* $occ_\pi(p) = \infty$. *Then*

1) $v\overline{w_m} = \overline{w_m}(mv)$,
2) $v\overline{w_p}^{\ -1} = \overline{w_p}^{\ -1}(\frac{1}{p}v)$.

Proof: By Theorem 3.10, we can choose k so large that $\overline{w_p}^{\ -1} = \overline{(\pi_1^{k-1})_1^k}$ and $v = \overline{(\pi_1^k h)_1^k} = \overline{(\pi_1^{k-1}(\frac{1}{p}h))_1^{k-1}}$. Also chose $n > k$ so that $\pi_{n-1}^n = w_p$ and so $\overline{w_p}^{\ -1} = \overline{(\pi_1^{n-1})_1^n}$.

To prove 1),

$$v\overline{w_m} = \overline{(\pi_1^k h)_1^k}\ \overline{(\pi_1^k w_m)_1^k} = \overline{(\pi_1^k h w_m)_1^k}$$

$$= \overline{(\pi_1^k w_m(mh))_1^k} = \overline{(\pi_1^k w_m)_1^k}\ \overline{(\pi_1^k(mh))_1^k} = \overline{w_m}(mv).$$

To prove 2), refer to the diagram in Figure 2. Choose $g : I_n \to I_n$ so that $h\pi_k^{n-1} = \pi_k^{n-1}g$. So $w_p h\pi_k^{n-1} = w_p\pi_k^{n-1}g$. But $w_p h\pi_k^{n-1} = \pi_k^n$ and $w_p h = (\frac{1}{p}h)w_p$, so $\pi_k^n g = (\frac{1}{p}h)w_p\pi_k^{n-1} = (\frac{1}{p}h)\pi_k^n$. Thus $\frac{1}{p}v = \overline{(\pi_1^k(\frac{1}{p}h)_1^k)} = \overline{(\pi_1^n g)_1^n}$.

Now we compute

$$\overline{w_p}^{\ -1}(\frac{1}{p}v) = \overline{(\pi_1^{n-1})_1^n}\ \overline{(\pi_1^n g)_1^n} = \overline{(\pi_1^{n-1}g)_1^n} = \overline{(\pi_1^k h\pi_k^{n-1})_1^n} = \overline{(\pi_1^k h)_1^k}\ \overline{(\pi_1^{n-1})_1^n} = v\overline{w_p}^{\ -1} \blacksquare$$

Now we can prove a structure theorem for the semigroup $\mathcal{OI}_\pi$ of induced open maps on K_π

3.16 Structure Theorem for $\mathcal{OI}_\pi$ *If K_π is even, then each $f \in \mathcal{OI}_\pi$ can be factored uniquely into the product $w_{\frac{a}{b}}u$, with $deg(f) = \frac{a}{b} \in Q_\pi$, $w_{\frac{a}{b}} \in \overline{\mathcal{W}_\pi^*}$ and $u \in \mathcal{HV}_\pi$. The rule for multiplication in $\mathcal{OI}_\pi$ is given by*

$$w_{\frac{a}{b}}u\ w_{\frac{c}{d}}v = w_{\frac{ac}{bd}}(\frac{c}{d}u)v.$$

If K_π is odd, then $\overline{\alpha}$ exists and each $f \in \mathcal{OI}_\pi$ can be factored uniquely into the product $\overline{\alpha}^i\ w_{\frac{a}{b}}u$, with $i \in \{0,1\} = \mathbb{Z}_2$, $deg(f) = \frac{a}{b} \in Q_\pi$, $w_{\frac{a}{b}} \in \overline{\mathcal{W}_\pi^*}$ and $u \in \mathcal{HV}_\pi$. The rule for multiplication is given by

$$\overline{\alpha}^i w_{\frac{a}{b}}u\ \overline{\alpha}^j w_{\frac{c}{d}}v = \overline{\alpha}^{i+nj}\ w_{\frac{ac}{bd}}\ (\frac{c}{d}(\overline{\alpha}^j u\overline{\alpha}^j))\ v.$$

Proof: Let $g \in \mathcal{OI}_\pi$. Then by Lemma 3.6, $g = \overline{\alpha}^i\ q\ u$, where $i \in \{0,1\}$, $q = \overline{(w_n)_1^k}$ for some positive integers n and k, and $u \in \mathcal{HV}_\pi$. By Corollary 3.11, $n = st$ for some positive integer t, where $\pi_1^k = w_{sf}$ is as defined in 3.11, and $\overline{(w_n)_1^k} = \overline{w_t}\ \overline{w_f}^{-1}$. Let $a = \frac{t}{gcd(t,f)}$ and $b = \frac{f}{gcd(t,f)}$. Then $q = \overline{w_{a\ gcd(t,f)}}\ \overline{w_{b\ gcd(t,f)}}^{-1} = \overline{w_a}\ \overline{w_b}^{-1}$. By Theorem 3.13, $deg(g) = \frac{a}{b} \in Q_\pi$.

To prove the uniqueness of the factorization, suppose $g = \overline{\alpha}^j\ w_{\frac{a'}{b'}}\ v$ is also a factorization of g. We consider two cases.

Case i) π is an even sequence. Then $\overline{\alpha}$ does not exist by Lemma 3.4, and so $i = j = 0$. Since $deg(g) = \frac{a'}{b'} = \frac{a}{b}$, we can assume that $gcd(a', b') = 1$ and so $\overline{w_a} = \overline{w_{a'}}$ and $\overline{w_b} = \overline{w_{b'}}^{-1}$. Hence $\overline{w_a}\ u = \overline{w_a}\ v$. Now we can choose k so large that $u = \overline{(\pi_1^k\ h_1)_1^k}$ and $v = \overline{(\pi_1^k\ h_2)_1^k}$ for some $h_1, h_2 \in \mathcal{H}^+$. But then $\overline{w_a}\ u = \overline{(\pi_1^k w_a h_1)_1^k}$ and $\overline{w_a}\ v = \overline{(\pi_1^k w_a h_2)_1^k}$. So by Lemma 3.3, $\pi_1^k w_a h_1 = \pi_1^k w_a h_2$ and so $h_1 = h_2$ by Lemma 2.6. Thus $u = v$, and Case i) is proved.

Case ii) π is an odd sequence. Then by Lemma 3.4, $\overline{\alpha}$ does exist, and all the coordinate maps of g are order preserving or all are order reversing. In the first case, $i = j = 0$, and in the second case $i = j = 1$. If $i = 0$, use the same argument as in Case i). If $i = 1$, multiply by $\overline{\alpha}$ and use the same argument as in Case i).

This completes the proof of the uniqueness of the factorization.

The rule for multiplication for the case of π even follows from Theorem 3.14 and Lemma 3.15 (part 2 is used repeatedly). Thus,

$$w_{\frac{a}{b}} u\ w_{\frac{c}{d}} v = w_{\frac{a}{b}} u\ \overline{w_c}\ \overline{w_d}^{\ -1} v = w_{\frac{a}{b}} \overline{w_c}\ (cu) \overline{w_d}^{\ -1} v = w_{\frac{ac}{b}}\ \overline{w_d}^{\ -1} (\frac{1}{d}(cu)) v = w_{\frac{ac}{bd}} (\frac{c}{d} u) v$$

Note that the assumption that π is even was not used in the calculations above, so we know the rule for multiplication in the case π is odd holds when $i = j = 0$. The general rule for the case of π odd is established using this and using additionally these properties of $\overline{\alpha}$, which follow from Theorem 3.5.

1) $\overline{\alpha}^j = \overline{\alpha}^{\ -j}$, 2) if b is odd, then $\overline{\alpha}^j\ \overline{w_b} = \overline{w_b}\ \overline{\alpha}^j$, and 3) $\overline{\alpha}^i\ \overline{w_a}\ \overline{\alpha}^j = \overline{\alpha}^{(i+aj) mod 2}\ \overline{w_a}$.

$$\overline{\alpha}^i\ w_{\frac{a}{b}} u\ \overline{\alpha}^j\ w_{\frac{c}{d}} v = \overline{\alpha}^i\ \overline{w_a}\ \overline{w_b}^{\ -1} u\ \overline{\alpha}^j\ w_{\frac{c}{d}} v = \overline{\alpha}^i\ \overline{w_a}\ \overline{w_b}^{\ -1} \overline{\alpha}^{\ -j} \overline{\alpha}^{\ j} u \overline{\alpha}^{\ j}\ w_{\frac{c}{d}} v =$$

$$\overline{\alpha}^i\ \overline{w_a}\ (\overline{\alpha}^j \overline{w_b})^{-1} \overline{\alpha}^j u \overline{\alpha}^j\ w_{\frac{c}{d}} v = \overline{\alpha}^i\ \overline{w_a}\ (\overline{w_b} \overline{\alpha}^j)^{-1} \overline{\alpha}^j u \overline{\alpha}^j\ w_{\frac{c}{d}} v = \overline{\alpha}^i\ \overline{w_a}\ \overline{\alpha}^{\ j} \overline{w_b}^{\ -1}\ \overline{\alpha}^j u \overline{\alpha}^j\ w_{\frac{c}{d}} v =$$

$$\overline{\alpha}^{(i+aj) mod 2} \overline{w_a}\ \overline{w_b}^{\ -1} \overline{\alpha}^j u \overline{\alpha}^j\ w_{\frac{c}{d}} v = \overline{\alpha}^{(i+aj) mod 2} w_{\frac{ac}{bd}} (\frac{c}{d} \overline{\alpha}^j u \overline{\alpha}^j) v \blacksquare$$

3.17 Corollary *Each open induced map $f : K_\pi \to K_\pi$ is no more than n to 1, where n is the numerator of the degree of f reduced to lowest terms.*

Proof: By Theorem 3.16, $f = \overline{\alpha}^i\ \overline{w_n}\ \overline{w_m}^{\ -1} u$. All of the factors are 1-1 maps, except $\overline{w_n}$, so for any $x \in K_\pi$, the cardinality of $f^{-1}(x)$ is the same as the cardinality of A_x, the set of points $y \in K_\pi$ such that $\overline{w_n}(y) = x$. If $card(A_x) > n$ for some $x \in K_\pi$, then in some coordinate k, $card(\pi_k(A_x)) > n$. But $w_n(\pi_k(A_x)) = x_k$ and w_n is at most n to 1, a contradiction. ■

4 Open maps on K_π

In this section, we will show that there are open maps on Knaster continua which are not induced, but that each open map is the uniform limit of induced open maps. Specifically, we construct an example of a homeomorphism on K_2 that is not induced. We also show that each open map $f \in \mathcal{O}_\rho^\pi$ is a uniform limit of induced open maps. In addition, we show that Debski's degree function $deg\colon \mathcal{O}_\rho^\pi \to Q^+$ is continuous.

Throughout the section, if $f, g\colon X \to I$ are maps on a compact space X, then $|f - g|$ denotes the distance from f to g in the 'sup' metric, that is,

$$|f - g| = sup\{|f(x) - g(x)|\colon\ x \in X\}.$$

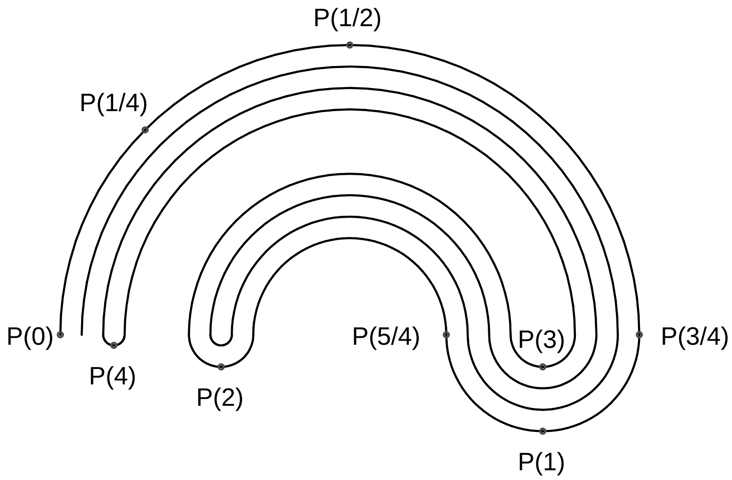
Also if $f, g : K_\pi \to K_\rho$, then $|f-g|$ denotes the distance from f to g in the 'sup' metric, that is,

$$|f-g| = sup\left\{\sum_{i=1}^{\infty} \frac{|\pi_i f(x) - \pi_i g(x)|}{2^i} \ : \ x \in K_\pi\right\}$$

An Example Let B be the standard bucket handle continuum constructed as a union of semicircles (see [8], p 205) situated in the (r,θ) plane so that the endpoint of B is the point $(1,\pi)$ and the semicircle containing the endpoint is the upper half of the unit circle, centered at the origin. Define B^* to be the visible composant of B. Note that B^* is comprised of a sequence, Q_i, of quarter-circles joined end to end. Denote the center of Q_i by c_i. We will define a continuous bijection $p : [0,\infty) \to B^*$. First define $p(0)$ to be the endpoint of B^* and $p(1/4)$ to be the midpoint of the first quarter circle Q_1. Next $p(1/2) = (1,\pi/2)$, the other endpoint Q_1. For $i > 1$, define $p(i/4)$ to be the first endpoint of Q_i in the natural ordering of B^*. Now extend p to all of $[0,\infty)$ as follows:

$$p(t) = \begin{cases} \text{the point } q \in Q_1 \text{ s.t. } \frac{\angle(p(0)c_1 q)}{(\pi/2)} = \frac{t}{1/2} & \text{for } 0 < t < 1/2 \\ \text{the point } q \in Q_i \text{ s.t. } \frac{\angle(p(i/4)c_i q)}{(\pi/2)} = \frac{t-i/4}{1/4} & \text{for } i > 1 \text{ and } i/4 < t < (i+1)/4 \end{cases}$$

The diagram below shows the first portion of B^*.



Now for each k, let $J_k = p([0, 2^k]) \subset B^*$, and for $l < k$ define the bonding map $f_l^k : J_k \to J_l$ by $f_l^k = f_l^{l+1} \cdots f_{k-1}^k$ for $l < k-1$. For $l = k-1$,

$$f_{k-1}^k(p(t)) = \begin{cases} p(t) & \text{if } 0 \leq t \leq 2^{k-1} \\ p(2^{k-1} - (t - 2^{k-1})) & \text{if } 2^{k-1} \leq t \leq 2^k - \frac{1}{2} \\ p(\frac{1}{2} + \frac{1-2^k}{2^{k-1}}(t - (2^k - \frac{1}{2})) & \text{if } 2^k - \frac{1}{2} \leq t \leq 2^k - \frac{1}{4} \\ p(\frac{-1}{2^{k-1}}(t - 2^k)) & \text{if } 2^k - \frac{1}{4} \leq t \leq 2^k \end{cases}$$

Denote the inverse limit of the arcs J_k and maps f_l^k by W_2. Note that each bonding map has degree 2, so W_2 is homeomorphic with K_2.

These definitions of J_k and f_l^k were constructed to satisfy the conditions of the Anderson-Choquet embedding theorem (See p 23 in [9]), and so the mapping $h\colon W_2 \to B$ given by $h((x_i)_{i=1}^\infty) = \lim_{i\to\infty} x_i$ is a homeomorphism.

Now define a homeomorphism $F\colon B \to B$ as follows:

$$F(r, \theta) = \begin{cases} (r, r\theta) & \text{if } 0 \leq \theta \leq \frac{\pi}{2},\ \frac{1}{3} \leq r \leq 1 \\ (r, r\frac{\pi}{2} + (2-r)(\theta - \frac{\pi}{2})) & \text{if } \frac{\pi}{2} \leq \theta \leq \pi,\ \frac{1}{3} \leq r \leq 1 \\ (r, \theta) & \text{otherwise} \end{cases}$$

4.1 Theorem *The homeomorphism $G = h^{-1}Fh\colon W_2 \to W_2$ is not induced.*

Proof: Consider the subset $X = \pi_1^{-1}(p(\frac{1}{2}))$ of W_2. Note that X is homeomorphic with the Cantor set, and hence is uncountable. The homeomorphism h carries X to the set Y consisting of all $(r, \frac{\pi}{2}) \in B$, and applying F to Y yields the set Z of all $(r, r\frac{\pi}{2}) \in B$. Now the map $\pi_1 h^{-1}$ takes each point $(r, r\frac{\pi}{2}) \in Z$ to the point $(1, r\frac{\pi}{2})$ and so we conclude that $\pi_1(G(X))$ is uncountable. But if G is induced by a sequence $g_l : J_{k_l} \to J_l$ of open maps, then $\pi_1(G(X)) = g_1(\pi_{k_1}(X))$ is finite, since $\pi_{k_1}(X) = \pi_{k_1}(\pi_1^{-1}(p(\frac{1}{2}))) = (f_1^{k_1})^{-1}(p(\frac{1}{2}))$ is finite. ■

The manner in which the homeomorphism G is defined on W_2 could be duplicated on any Knaster continuum, because they can be embedded in the plane in the same manner as W_2 (see Watkins [11]).

The open induced approximation theorem. The next three lemmas lead to a proof of theorem 4.7 : any open map from K_π to K_ρ can be approximated by an induced open map of the same degree.

4.2 Lemma *If $f, g \in \mathcal{O}$ and $|f - g| \leq \frac{1}{2}$, then*

1. *$deg(f) = deg(g)$*
2. *there is an order preserving homeomorphism, h, such that $f = gh$.*

Proof: Since $|f - g| \leq \frac{1}{2}$, we have that $f(0) = g(0)$. By Theorem 2.1, there are numbers $0 = a_0 < a_1 < ... < a_n = 1$ for which $f|_{[a_i, a_{i+1}]}$ is a homeomorphism onto I. For each i, let

$a_{i+\frac{1}{2}} = f|^{-1}_{[a_i,a_{i+1}]}(\frac{1}{2})$, $I_L = [0, \frac{1}{2}]$, and $I_U = [\frac{1}{2}, 1]$. Note that for each i, $f([a_{i+\frac{1}{2}}, a_{i+\frac{3}{2}}]) \subseteq I_L$ or $f([a_{i+\frac{1}{2}}, a_{i+\frac{3}{2}}]) \subseteq I_U$. Since $|f - g| \leq \frac{1}{2}$ and g is open, it follows that $g|^{-1}_{[a_i,a_{i+1}]}(\{0,1\})$ is a singleton for each i, which makes $deg(g) \leq deg(f)$. Similiarly, $deg(f) \leq deg(g)$. Next, using Theorem 2.4, write $f = \alpha^i w_n h_1$ and $g = \alpha^i w_n h_2$ for some order preserving homeomorphisms h_1 and h_2. Let $h = h_2^{-1} h_1$. Then $f = gh$. ■

4.3 Lemma *If $f, g \in \mathcal{O}^+$ with $f(0) = g(0)$ and $|w_n\, f - w_n\, g| < \frac{1}{2}$, then for any i and any $t \in I$, the interval between $f(t)$ and $g(t)$ cannot contain both $\frac{i}{n}$ and $\frac{i+1}{n}$.*

Proof. Suppose the lemma is false. Let t_1 be the smallest t which violates the lemma and let i be the smallest i such that $\frac{i}{n}$ and $\frac{i+1}{n}$ both lie between $f(t_1)$ and $g(t_1)$. We may assume that $f(t_1)$ is less than $g(t_1)$. Now $t_1 > 0$, for otherwise $f(t_1) = g(t_1)$, since $f(0) = g(0)$. Further, either $f(t_1) = \frac{i}{n}$ or $g(t_1) = \frac{i+1}{n}$, for otherwise $f(t_1) < \frac{i}{n} < \frac{i+1}{n} < g(t_1)$ and by the continuity of f and g, there is a $t < t_1$ such that $f(t) < \frac{i}{n} < \frac{i+1}{n} < g(t)$, a violation of the choice of t_1.

Case i. $f(t_1) = \frac{i}{n}$. In this case, $g(t_1) > \frac{i+1}{n}$, otherwise $g(t_1) = \frac{i+1}{n}$ and so $|w_n g(t_1) - w_n f(t_1)| = 1$, a violation of the assumption that $|w_n g - w_n f| < \frac{1}{2}$. Also, $g(t_1) \leq \frac{i+2}{n}$, otherwise by the continuity of f and g there is a $t < t_1$ such that $f(t) < \frac{i+1}{n} < \frac{i+2}{n} < g(t)$, a violation of the choice of t_1.

Now as t decreases from t_1, $f(t)$ must increase by the minimality of t_1. Further, since f is open, $f(t)$ must continue to increase until it reaches 1. Let $t' = \max\{t \in [0, t_1] | f(t) = \frac{i+1}{n}\}$. Likewise, since g is open, as t decreases from t_1, $g(t)$ must either increase from $g(t_1)$ to 1 or decrease from $g(t_1)$ to 0.

Subcase 1: $g(t)$ increases to 1. Let $t'' = \max\{t \in [0, t_1] | g(t) = \frac{i+2}{n}\}$. First note that $t' < t''$ is false. For otherwise $\frac{i+1}{n}$ and $\frac{i+2}{n}$ lie between $f(t'')$ and $g(t'')$, a violation of the choice of t_1. So $t' \geq t''$. Then $\frac{i+1}{n} = f(t') < g(t_1) < g(t') \leq g(t'') = \frac{i+2}{n}$. Hence

(**) $|w_n(f(t')) - w_n(g(t_1))| < |w_n(f(t')) - w_n(g(t'))|$. But the left hand side of (**) is greater than 1/2 since $|w_n(f(t_1)) - w_n(g(t_1))| < \frac{1}{2}$ and $|w_n(f(t')) - w_n(f(t_1))| = 1$. So the right hand side of (**) is greater than 1/2, in contradiction to the hypothesis $|w_n f - w_n g| < \frac{1}{2}$. So subcase 1 cannot occur.

Subcase 2: $g(t)$ decreases to 0. Let $t'' = \sup\{t \in [0, t_1] | g(t) = 0 \text{ or } f(t) = 1\}$.

By the continuity of f and g, $f(t'') = 1$ or $g(t'') = 0$.

Suppose $f(t'') = 1$. Then $\frac{i+2}{n} = 1$ (otherwise $\frac{i+2}{n}$ and $\frac{i+3}{n}$ lie between $g(t'')$ and $f(t'')$, violating the choice of t_1). Also $g(t'') > \frac{i+1}{n}$ for the same reason. Note that $g(t') \in [g(t''), f(t'')] \subset [\frac{i}{n}, \frac{i+1}{n}]$ and so

$|w_n(g(t')) - w_n(f(t''))| < |w_n(g(t'')) - w_n(f(t''))| < \frac{1}{2}$.

Now $|w_n(f(t')) - w_n(g(t'))| < \frac{1}{2}$ and so, by the triangle inequality, we have $|w_n(f(t'')) - w_n(f(t'))| < 1$, which is false since $f(t'') = \frac{i+2}{n}$ and $f(t') = \frac{i+1}{n}$. Thus $f(t'') \neq 1$.

Hence $f(t'') < 1$ and it must be that $g(t'') = 0$. Now since $f(t'') \geq \frac{i}{n}$, we have that $\frac{i}{n} = 0$ (otherwise $\frac{i-1}{n}$ and $\frac{i}{n}$ lie between $g(t'')$ and $f(t'')$, violating the choice of t_1). Also $f(t'') < \frac{i+1}{n}$ for the same reason. Now let $t''' = \max\{t \in [0, t_1] | g(t) = \frac{i+1}{n}\}$. Note that $t''' \in [t'', t_1]$ and so $|w_n(f(t''')) - w_n(g(t''))| < \frac{1}{2}$. Now $|w_n(f(t''')) - w_n(g(t'''))| < \frac{1}{2}$ and so, by the triangle inequality, we have $|w_n(g(t'')) - w_n(g(t'''))| < 1$, which is false since $g(t'') = \frac{i}{n}$ and $g(t''') = \frac{i+1}{n}$. Hence $g(t'') > 0$. So subcase 2 cannot occur either.

So Case i. cannot occur.

Case ii. $g(t_1) = \frac{i+1}{n}$. This case is similar to case i. First show that $\frac{i-1}{n} \leq f(t_1) < \frac{i}{n}$ and $g(t)$ increases as t decreases from t_1. Then show there are two subcases:

Subcase 1: $f(t)$ increases to 1. This subcase is eliminated in a manner similar to the manner Subcase 2 of Case i is eliminated.

Subcase 2: $f(t)$ decreases to 0. This subcase is eliminated in a manner similar to the manner Subcase 1 of Case i is eliminated.

In this manner it is shown that Case ii cannot occur either. ■

4.4 Lemma *If $n > 1$ is a positive integer, $f, g \in \mathcal{O}$ with $f(0) = g(0)$, and $|w_n f - w_n g| < \frac{1}{2}$ then $|f - g| < |w_n f - w_n g|$.*

Proof. Choose a t_1 so that $|f - g| = |f(t_1) - g(t_1)|$. Without loss of generality, assume that $f(t_1) < g(t_1)$. Now $0 < t_1$ and $t_1 < 1$. For the moment assume that $0 < f(t_1) < 1$ and $0 < g(t_1) < 1$. Then since f, g are open and $|f(t_1) - g(t_1)|$ is maximum, as t increases (or decreases) from t_1, $f(t)$ and $g(t)$ must both increase or both decrease. For if $f(t)$ increases and $g(t)$ decreases as t increases, say, then allowing t to decrease from t_1 will cause $f(t)$ to decrease and $g(t)$ to increase. In one direction or the other, $|f(t) - g(t)|$ must increase, a contradiction, since $|f(t_1) - g(t_1)|$ is maximum.

By Lemma 4.3, there is at most one $\frac{i}{n}$ between $f(t_1)$ and $g(t_1)$. If there is no $\frac{i}{n}$ strictly between $f(t_1)$, $g(t_1)$ then $|w_n f(t_1) - w_n g(t_1)| = n\ |f(t) - g(t)|$. If there is one, say $\frac{i}{n}$, we consider three cases.

Case 1. $w_n f(t_1)$and $w_n g(t_1)$ are between 0 and $\frac{1}{2}$.

Suppose $\frac{i}{n}$ is between $f(t_1)$ and $g(t_1)$. Then $w_n(\frac{i}{n}) = 0$. For suppose $w_n(\frac{i}{n}) = 1$. Then by either increasing t from t_1 or decreasing t from t_1 , $w_n g(t)$ stays below 1/2 until $w_n f(t)$ decreases to 0, at which point Lemma 1 is violated. Without loss of generality assume that $\frac{i}{n} - f(t_1) \leq g(t_1) - \frac{i}{n}$. Then we note from the geometry that $g(t_1) - f(t_1) \leq w_n g(t_1)$. Now as t increases or decreases from t_1, $w_n f(t)$ decreases to 0 before $w_n g(t)$ increases to $\frac{1}{2}$ and hence at that point $g(t_1) - f(t_1) \leq w_n g(t) - w_n f(t)$ and the lemma holds.

Case 2. $w_n f(t_1)$ and $w_n g(t_1)$ are between $\frac{1}{2}$ and 1,

This case is nearly identical to Case 1.

Case 3. $w_n f(t_1)$ is between 0 and $\frac{1}{2}$ and $w_n g(t_1)$ is between $\frac{1}{2}$ and 1.

This case cannot occur. For by increasing or decreasing t from t_1, we can decrease $w_n f(t)$ to 0 before $w_n g(t)$ increases to 1, at which point the distance from $w_n g$ to $w_n f$ exceeds $\frac{1}{2}$, a contradiction. ■

In [6], Debski defines an approximating sequence as follows:

Let $f : K_\pi \to I$ be an open map. A sequence of open maps $f_i : I_i \to I$ is called an **approximating sequence for** f provided that the sequence $f_i \pi_i : K_\pi \to I$ converges to f in the uniform metric.

He then proves an approximation theorem [6, p24]:

4.5 Debski's Approximation Theorem *Every open map $f : K_\pi \to I$ has an approximating sequence f_i. Furthermore, for sufficiently large i, the sequence $\frac{deg(f_i)}{deg(\pi_1^i)}$ is constant.*

Let $\overline{0}$ denote the point in K_π all of whose coordinates are 0.

4.6 Corollary *Let $f \in \mathcal{O}_\rho^\pi$. If ρ is even then $f(\overline{0}) = \overline{0}$, and if ρ is odd (with no twos) then $f(\overline{0}) = \overline{0}$ or $\overline{\alpha}(f(\overline{0})) = \overline{0}$.*

Proof: For each positive integer k, let $f_i : I_i \to I_k$ be an approximating sequence for the kth coordinate map of f, $\rho_k f : K_\pi \to I$. Since the maps f_i are open, we have $f_i(\pi_i(\overline{0})) = f_i(0)$ is 0 or 1. Since the sequence $f_i\pi_i$ converges uniformly to $\rho_k f$, we know $f_i(\pi_i(\overline{0}))$ converges to $\rho_k(f(\overline{0}))$, so $\rho_k(f(\overline{0}))$ is 0 or 1. Now if ρ is even, then for each k there is an $l > k$ such that $\rho_k^l = w_{2m}$ for some m. Hence $\rho_k(f(\overline{0})) = \rho_k^l(\rho_l(f(\overline{0}))) = w_{2m}(j) = 0$, where j is 0 or 1. This shows that $f(\overline{0}) = \overline{0}$ if ρ is even. If ρ is odd, let $j = \rho_1(f(\overline{0}))$. Now if for some k, $\rho_k(f(\overline{0})) = t \neq j$, then $j = \rho_1(f(\overline{0})) = \rho_1^k(\rho_k(f(\overline{0}))) = \rho_l^k(t) = t \neq j$. This shows that $f(\overline{0}) = \overline{0}$ or $\overline{\alpha} f(\overline{0}) = \overline{0}$. ■

Debski defines the **degree of an open map** $f : K_\pi \to I$ to be the constant guaranteed by 4.5. The **degree of an open map** $f\colon K_\pi \to K_\rho$ is defined as the degree of $\rho_1 f$. Note that this definition extends the notion of the degree of an induced open map from K_π to K_ρ.

4.7 Theorem *If $f : K_\pi \to K_\rho$ is an open map and $\epsilon > 0$, then there is an open induced map $g : K_\pi \to K_\rho$ such that $|f - g| < \epsilon$ and $deg(f) = deg(g)$.*

Proof:

Without loss of generality, we can assume that $f(\overline{0}) = \overline{0}$. Hence for each j, $\rho_j(f(\overline{0})) = 0$ and so the terms of any approximating sequence for $\rho_j f$ can be assumed to take 0 to 0. We make this assumption below. Also, we assume that $\epsilon < 1/2$. For each $j \in \mathbb{N}$, there is a $\delta_j < \epsilon/4$ such that

1) if $|x - y| < \delta_j$, then $|\rho_1^j(x) - \rho_1^j(y)| < \epsilon/4$.

For each $j \in \mathbb{N}$, let $\{f_k^j : I_k \to I_j\}$ be an approximating sequence for $\rho_j f$. Choose N_1 so that the following two conditions are met:

2) $|f_{N_1}^1 \pi_{N_1} - \rho_1 f| < \epsilon/4$.

3) for $N > N_1$, $\frac{deg(f_N^1)}{deg(\pi_1^N)} = \frac{deg(f_{N_1}^1)}{deg(\pi_1^{N_1})}$. (Note: this number is actually $deg(f)$. See [6])

For $k > 1$, choose $N_k > N_{k-1}$ such that the following two conditions are met:

4) $|f_{N_k}^k \pi_{N_k} - \rho_k f| < \delta_k$

5) For $N > N_k$, $\frac{deg(f_N^k)}{deg(\pi_1^N)} = \frac{deg(f_{N_k}^k)}{deg(\pi_1^{N_k})}$.

Claim 1. For each k, $|f_{N_1}^1 \pi_{N_1}^{N_k} - \rho_1^k f_{N_k}^k| < \epsilon/2$.

To see this note that

$$|f_{N_1}^1 \pi_{N_1}^{N_k} \pi_{N_k} - \rho_1^k f_{N_k}^k \pi_{N_k}| = |f_{N_1}^1 \pi_{N_1} - \rho_1^k f_{N_k}^k \pi_{N_k}| \leq$$

$$|f_{N_1}^1 \pi_{N_1} - \rho_1 f| + |\rho_1 f - \rho_1^k f_{N_k}^k \pi_{N_k}| = |f_{N_1}^1 \pi_{N_1} - \rho_1 f| + |\rho_1^k \rho_k f - \rho_1^k f_{N_k}^k \pi_{N_k}|.$$

From 2), the first term of this sum is less than $\epsilon/4$, and since $|\rho_k f - f_{N_k}^k \pi_{N_k}| < \delta_k$, we know from 1) that the second term is less than or equal to $\epsilon/4$. So we have

$|f^1_{N_1}\pi_1^{N_k}\pi_{N_k} - \rho_1^k f^k_{N_k}\pi_{N_k}| < \epsilon/2$. Since π_{N_k} is a surjection, it can be cancelled from the right to yield Claim 1.

Since we have assumed that $\epsilon < 1/2$, Claim 1 shows that $|f^1_{N_1}\pi^{N_k}_{N_1} - \rho_1^k f^k_{N_k}| < 1/2$, for each positive integer k. So by Lemma 4.2, there is an order preserving homeomorphism h_k such that $f^1_{N_1}\pi^{N_k}_{N_1} = \rho^k_1 f^k_{N_k} h_k$. Let $g_1 = f^1_{N_1}$, and for each $k > 1$, let $g_k = f^k_{N_k} h_k$. Note that $g_k \in \mathcal{O}^+$ for all k. Now we have

$$(*) \quad \text{for each j,} \quad g_1\pi^{N_j}_{N_1} = \rho^j_1 g_j$$

Claim 2. For each k, $g_k\pi^{N_{k+1}}_{N_k} = \rho^{k+1}_k g_{k+1}$. Letting $j = k+1$ in equation (*) yields

$$\rho^k_1\rho^{k+1}_k g_{k+1} = \rho^{k+1}_1 g_{k+1} = g_1\pi^{N_{k+1}}_{N_1} = g_1\pi^{N_k}_{N_1}\pi^{N_{k+1}}_{N_k}.$$

Letting $j = k$ in equation (*) and multiplying both sides of the resulting equation on the right by $\pi^{N_{k+1}}_{N_k}$ gives

$$g_1\pi^{N_k}_{N_1}\pi^{N_{k+1}}_{N_k} = \rho^k_1 g_k \pi^{N_{k+1}}_{N_k},$$

and so

$$\rho^k_1\rho^{k+1}_k g_{k+1} = \rho^k_1 g_k \pi^{N_{k+1}}_{N_k}.$$

Since ρ^k_1 is a standard open map, Lemma 2.6 guarantees that ρ^k_1 can be cancelled on the left of this equation to yield $\rho^{k+1}_k g_{k+1} = g_k\pi^{N_{k+1}}_{N_k}$. This proves Claim 2.

Now by Claim 2 and Lemma 3.1, the sequence of maps g_k induces an open map $g : K_\pi \to K_\rho$. Since $g_1 = f^1_{N_1}$, we have that $deg(g) = \frac{deg(g_1)}{deg(\pi^{N_1}_1)} = \frac{deg(f^1_{N_1})}{deg(\pi^{N_1}_1)} = deg(f)$.

In order to show that $|f - g| < \epsilon$, we need to establish

Claim 3. For each k, $|g_k\pi_{N_k} - \rho_k f| < \epsilon$.

For $k = 1$, this follows from the definition of g_1 and condition 2) above. When $k > 1$, the triangle inequality gives $|g_k\pi_{N_k} - \rho_k f| \le |g_k\pi_{N_k} - f^k_{N_k}\pi_{N_k}| + |f^k_{N_k}\pi_{N_K} - \rho_k f|$. From 4) and $\delta_k < \epsilon/4$, the second term in this sum is less than $\epsilon/4$. To bound the first term, note that

$$|\rho^k_1 g_k - \rho^k_1 f_{N_k}| = |\rho^k_1 f^k_{N_K} h_k - \rho^k_1 f^k_{N_k}| = |f^1_{N_1}\pi^{N_k}_{N_1} - \rho^k_1 f^k_{N_k}| < \epsilon/2 < 1/2$$

(The first equality uses the definition of g_k, the second that of h_k; the first inequality is Claim 1, the second is by choice of ϵ.) Since these are interval maps and ρ^k_1 is a standard map, Lemma 4.4 yields $|g_k - f^k_{N_k}| < |\rho^k_1 g_k - \rho^k_1 f^k_{N_k}| < \epsilon/2$. Because π_{N_k} is a surjection, $|g_k\pi_{N_k} - f^k_{N_k}\pi_{N_k}| = |g_k - f^k_{N_k}| < \epsilon/2$. This bounds the first term, and establishes Claim 3.

To complete the argument, choose $x \in K_\pi$ so that $|f - g| = |f(x) - g(x)|$. Then $|f - g| = \sum_{k=1}^{\infty} |g_k\pi_{N_k}(x) - \rho_k f(x)| \cdot 2^{-k} < \sum_{k=1}^{\infty} \epsilon \cdot 2^{-k} = \epsilon \cdot \sum_{k=1}^{\infty} 2^{-k} = \epsilon$. ∎

4.8 Theorem *If f and g are open maps from K_π to K_ρ with $|f - g| < 1/4$, then $deg(f) = deg(g)$.*

Proof: In the case that f and g are both induced, there are f_i and g_i, sequences of inducing functions for f and g; furthermore, these sequences can be found so that for each i, $dom(f_i) = dom(g_i) = I_{k_i}$.

Now, for any $x_{k_1} \in I_{k_1}$, let $x \in \pi_{k_1}^{-1}(x_{k_1})$ and $x_k = \pi_k(x)$. Then,

$$\frac{1}{2}|f_1(x_{k_1})-g_1(x_{k_1})| \leq \frac{1}{2}|f_1(x_{k_1})-g_1(x_{k_1})|+\sum_{n=2}^{\infty} 2^{-n}|f_n(x_{k_n})-g_n(x_{k_n})| = |f(x)-g(x)| \leq |f-g| < \frac{1}{4}.$$

Thus, $|f_1 - g_1| \leq \frac{1}{2}$, and so f_1 and g_1 have the same degree. Finally, $deg(f) = \frac{deg(f_1)}{deg(\pi_1^{k_1})} = \frac{deg(g_1)}{deg(\pi_1^{k_1})} = deg(g)$.

When one of f or g is not induced, use Theorem 4.7 with $\epsilon = \frac{1}{4} - \frac{|f-g|}{2}$ to find induced maps f^* and g^* with the same degrees as f and g and so that $|f - f^*| < \epsilon$ and $|g - g^*| < \epsilon$.■

We have an immediate corollary.

4.9 Corollary *The decomposition of $\mathcal{O}_\rho^\pi$ into degree classes is an open decomposition and each class contains a dense set of induced open mappings. Further, the degree homomorphism $deg\colon \mathcal{O}_\pi \to Q_\pi$ is a continuous open mapping if $\mathcal{O}_\pi$ is given the sup metric and Q_π is given the discrete topology.*

For each rational $\frac{r}{s} \in Q_\pi$, let $\mathcal{O}_\pi(\frac{r}{s}) = deg^{-1}(\frac{r}{s})$, the open maps of degree $\frac{r}{s}$. Each element f of this degree class is a uniform limit of open induced maps from the class. For example, the degree one open maps are uniform limit of degree one induced open maps, the vertically induced homeomorphisms. It is natural to ask whether the degree one open maps must themselves be homeomorphisms. More generally, we can ask the following question.

4.10 Question *Suppose $\frac{r}{s}$ is invertible in Q_π. Are the members of $\mathcal{O}_\pi(\frac{r}{s})$ all homeomorphisms?*

We have produced an algebraic structure theorem for the semigroup $\mathcal{IO}_\pi$, Theorem 3.16. In view of this and Corollary 4.9, it is natural to seek a structure theorem for the semigroup $\mathcal{O}_\pi$. In particular, we ask the following question:

4.11 Question *Can each $f \in \mathcal{O}_\pi$ can be factored into $w_{\frac{r}{s}}\, u$, where u is a degree one open map of K_π?*

We can give partial answers to this question, and note that an affirmative answer to the first question implies an affirmative answer to the second question.

4.12 Lemma *If $\frac{r}{s}$ is invertible in Q_π, then $\mathcal{O}_\pi(\frac{r}{s}) = w_{\frac{r}{s}}\mathcal{O}_\pi(1)$.*

Proof: $f \in \mathcal{O}_\pi(\frac{r}{s})$ if and only if $f = w_{\frac{r}{s}}(w_{\frac{s}{r}} f) \in w_{\frac{r}{s}}\mathcal{O}_\pi(r)$. ■

4.13 Corollary *Let γ denote the sequence $2, 3, 2, 3, 5, \cdots$, of primes in which each prime occurs infinitely often. Then each open map $f \in \mathcal{O}_\gamma$ can be written uniquely as $w_{\frac{r}{s}}\ u$, where $\frac{r}{s}$ is the degree of f and u is a degree one open map.*

Proof: By Lemma 4.12, f can be written as claimed. To show uniqueness, suppose $f = w_{\frac{r}{s}}\ u = w_{\frac{r}{s}}\ v$. Then multiply on the left by $w_{\frac{s}{r}}$ and conclude that $u = v$. ■

4.14 Lemma *If $u, v \in \mathcal{HV}_\pi$ and $|\overline{w_r}\ u - \overline{w_r}\ v| < \frac{1}{2}$, then $|\overline{w_r}\ u - \overline{w_r}\ v| \geq |u - v|$.*

Proof: Let $\epsilon > 0$ be arbitrarily chosen. Choose n so large that $\sum_{i=n+1}^{\infty} \frac{1}{2^i} < \epsilon$ and we can choose homeomorphisms h, g of I so that $u = \overline{(\pi_1^n\ h)_1^n}$, and $v = \overline{(\pi_1^n\ g)_1^n}$. Then $\overline{w_r}\ u = \overline{(\pi_1^n\ w_r\ h)_1^n}$, and $\overline{w_r}\ v = \overline{(\pi_1^n\ w_r\ g)_1^n}$.

¿From the definition of distance in K_π and the given inequality, we have $|\pi_1^n\ w_r\ h - \pi_1^n\ w_r\ g| < \frac{1}{2}$, and so by Lemma 4.4 we have

$$|\pi_k^n\ w_r\ h - \pi_k^n\ w_r\ g| \geq |\pi_k^n\ h - \pi_k^n\ g|$$

for all k from 1 to n. Now let $x \in K_\pi$ so that $|u - v| = |u(x) - v(x)|$. It follows that

$$|\overline{w_r}\ u - \overline{w_r}\ v| \geq |\overline{w_r}\ u(x) - \overline{w_r}\ v(x)| = \sum_{i=1}^{\infty} \frac{|\pi_i\ \overline{w_r}\ u(x) - \pi_i\ \overline{w_r}\ v(x)|}{2^i}$$

$$\geq \sum_{i=1}^{n} \frac{|\pi_i\ \overline{w_r}\ u(x) - \pi_i\ \overline{w_r}\ v(x)|}{2^i} = \sum_{i=1}^{n} \frac{|\pi_i^n\ w_r\ h(x_n) - \pi_k^n\ w_r\ g(x_n)|}{2^i}$$

$$\geq \sum_{i=1}^{n} \frac{|\pi_i^n\ h(x_n) - \pi_k^n\ g(x_n)|}{2^i} = \sum_{i=1}^{n} |\pi_i u(x) - \pi_i v(x)| \geq |u - v| - \epsilon.$$

Since ϵ was arbitrary, the lemma follows. ■

The following theorem is the closest we have come to a factorization theorem for $\mathcal{O}_\pi$.

4.15 Theorem *If $deg(f) = \frac{r}{s}$, then $f = w_{\frac{r}{s}}\ u$ for some continous surjection u of K_π.*

Proof: By Theorem 4.7, $f = \lim_{n\to\infty} \overline{w_r}\ u_n$, where $u_n \in \mathcal{HV}_\pi$ for each n. By Lemma 4.14, $|u_n - u_m| < |\overline{w_r}\ u_n - \overline{w_r}\ u_m|$ for sufficiently large n, m and so u_n is a Cauchy sequence. Since the space of continuous maps from K_π to K_π is complete, the sequence u_n converges uniformly to a continuous surjection $u : K_\pi \to K_\pi$. But also composition of functions is a continuous operation on the space of continuous maps of K_π, so the sequence $\overline{w_r}\ u_n$ converges to $\overline{w_r}\ u$. Hence $f = \overline{w_r}\ u$. ■

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