MA 214 Calculus IV (Spring 2016)

Section 2

Homework Assignment 4

Solutions

1. Boyce and DiPrima, p. 90, Problem 16.

   Solution: (a) A sketch of the graph of $f(y) = ry\ln(K/y)$ for $r = 2$ and $K = 4$ is given in Figure 1. The critical points are $y_1 = 0$ and $y_2 = K$, which divide $(0, \infty)$ into two intervals, namely $(0, K)$ and (K, ∞), on which $y' > 0$ and $y' < 0$, respectively. Hence the equilibrium solution $y = 0$ is unstable, and $y = K$ is asymptotically stable.

   (b) We have $f'(y) = r(\ln(K/y) - 1)$, and $f'(y) = 0$ if and only if $y = K/e$. Since $y'' = f'(y)f(y) > 0$ on $(0, K/e)$, $y'' < 0$ on $(K/e, K)$, and $y'' > 0$ on (K, ∞), the graph of y versus t is concave up on $(0, K/e)$, concave down on $(K/e, K)$, and concave up on (K, ∞).

   (c) Let $g(y) = \ln y$ and consider a specific $y < K$. By the mean value theorem, there exists an $x \in (y, K)$ such that $g'(x) = \frac{\ln K - \ln y}{K - y}$. Since $g'(x) = 1/x > 1/K$, we have $\frac{\ln K - \ln y}{K - y} > \frac{1}{K}$ or $\ln(K/y) > 1 - y/K$. Hence for $0 < y \le K$, we have $f(y) = ry\ln(K/y) \ge ry(1 - y/K)$.

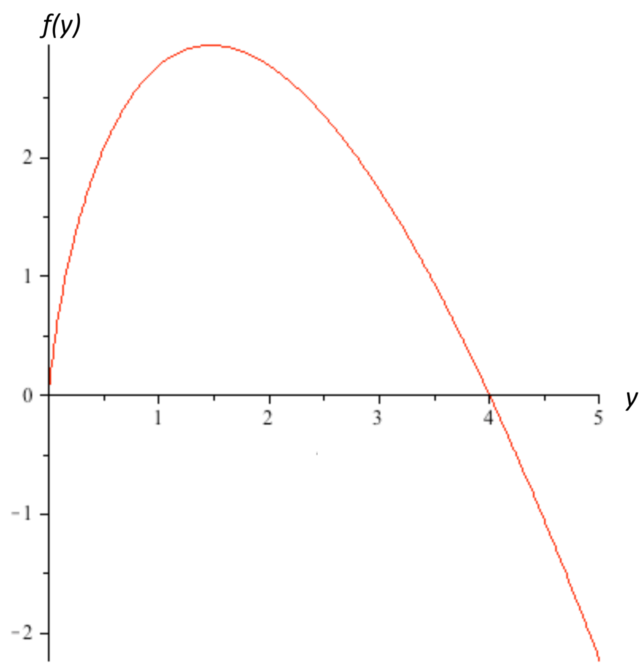

Figure 1: Sketch of the graph of $f(y) = ry\ln(K/y)$ for $r = 2$ and $K = 4$.

2. Boyce and DiPrima, p. 91, Problem 20.

Solution: (a) The Schaefer model is defined by the differential equation $y' = f(y) = r(1 - y/K)y - Ey$. The function $f(y)$ can be recast as

$$f(y) = \frac{ry}{K}\left[K\left(1 - \frac{E}{r}\right) - y\right].$$

The equation $f(y) = 0$ has two solutions $y_1 = 0$ and $y_2 = K(1 - E/r)$. Note that $y_2 > 0$ if $E < r$, in which case y_1 and y_2 are the equilibrium solutions of the Schaefer model.

(b) The critical points $y_1 = 0$ and $y_2 = K(1 - E/r)$ divide $(0, \infty)$ into two open intervals, namely $(0, y_2)$ and (y_2, ∞), on which $y' > 0$ and $y' < 0$, respectively. Hence the equilibrium point $y_1 = 0$ is unstable, and $y_2 = K(1 - E/r)$ is asymptotically stable.

(c) By definition, $Y := Ey_2 = EK(1 - E/r)$, which is a quadratic function in E.

(d) Setting $dY/dE = 0$, we find $E = r/2$, at which Y assumes its maximum value $Y_m = (rK/2)(1 - 1/2) = rK/4$.

3. Boyce and DiPrima, p. 101, Problem 4.

Solution: Let $M = 2xy^2 + 2y$ and $N = 2x^2y + 2x$. Since $\partial M/\partial y = \partial N/\partial x = 4xy + 2$, the give differential equation is exact. Integrating both sides of the equation

$$\frac{\partial \psi}{\partial x} = 2xy^2 + 2y$$

with respect to x while keeping y fixed, we obtain

$$\psi(x, y) = x^2y^2 + 2xy + g(y),$$

which implies

$$\frac{\partial \psi}{\partial y} = 2x^2y + 2x + g'(y).$$

Equating $\partial\psi/\partial y$ to N, we conclude that $g'(y) = 0$. Thus $g = C_1$ for some constant C_1. Without loss of generality, we take $C_1 = 0$. Hence solutions of the given differential equation are given implicitly by

$$x^2y^2 + 2xy = C,$$

where C is an arbitrary constant.

4. Boyce and DiPrima, p. 101, Problem 8.

Solution: Let $M(x, y) = e^x \sin y + 3y$, $N(x, y) = -(3x - e^x \sin y)$. Then

$$\frac{\partial M}{\partial y} = e^x \cos y + 3, \qquad \frac{\partial N}{\partial x} = -3 + e^x \sin y.$$

Since $\partial M/\partial y \neq \partial N/\partial x$, the given equation is not exact.

5. Boyce and DiPrima, p. 101, Problem 16.

Solution: Let $M = y\,e^{2xy} + x$ and $N = bx\,e^{2xy}$. Then we have

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xy\,e^{2xy}, \quad \frac{\partial N}{\partial x} = b\left(e^{2xy} + 2xy\,e^{2xy}\right).$$

For the given differential equation to be exact, we must have $b = 1$.

Henceforth we take $b = 1$. Integrating both sides of the equation

$$\frac{\partial \psi}{\partial y} = xe^{2xy}$$

with respect to y while keeping x fixed, we obtain

$$\psi(x, y) = \frac{1}{2}e^{2xy} + f(x),$$

which implies

$$\frac{\partial \psi}{\partial x} = ye^{2xy} + f'(x).$$

Equating $\partial\psi/\partial x$ to M, we see that $f'(x) = x$ and thus $f(x) = x^2/2 + C_1$ for some constant C_1. Without loss of generality, we take $C_1 = 0$. Hence solutions of the exact differential equation with $b = 1$ are given by

$$e^{2xy} + x^2 = C,$$

where C is an arbitrary constant.

6. Boyce and DiPrima, p. 101, Problem 21.

Solution: Let $M = y$ and $N = 2x - ye^y$. Then $\partial M/\partial y = 1$ and $\partial N/\partial x = 2$. Hence the given equation is not exact. The differential equation becomes exact after we multiply both sides of the equation by the integrating factor $\mu = y$, because

$$\frac{\partial}{\partial y}(\mu M) = 2y, \quad \frac{\partial}{\partial x}(\mu N) = 2y.$$

Integrating both sides of the equation

$$\frac{\partial \psi}{\partial x} = y^2$$

with respect to x while keeping y fixed, we obtain

$$\psi(x, y) = y^2 x + g(y),$$

which implies

$$\frac{\partial \psi}{\partial y} = 2xy + g'(x).$$

Equating $\partial\psi/\partial y$ to $N = 2xy - y^2x^y$, we get $g'(y) = -y^2e^y$, the integration of which gives

$$g(y) = -y^2e^y + 2ye^y - 2e^y + C_1$$

for some constant C_1. Without loss of generality, we take $C_1 = 0$. Hence solutions of the given differential equation are given implicitly by

$$y^2x - (y^2 - 2y + 2)e^y = C,$$

where C is an arbitrary constant.

7. Boyce and DiPrima, p. 102, Problem 28.

Solution: We seek an integrating factor μ which satisfies

$$\frac{\partial}{\partial y}(\mu y) = \frac{\partial}{\partial x}\left(\mu(2xy - e^{-2y})\right)$$

or

$$\frac{\partial\mu}{\partial y}y + \mu = \frac{\partial\mu}{\partial x}\left(2xy - e^{-2y}\right) + 2\mu y.$$

By inspection, we see that there exists an integrating factor $\mu = \mu(y)$ which satisfies the equation

$$\frac{d\mu}{dy}y + \mu = 2\mu y.$$

For $y \neq 0$, we recast the preceding equation as

$$\frac{d\mu}{dy} + \frac{1-2y}{y}\mu = 0.$$

An integrating factor of this linear first-order ODE is:

$$\nu(y) = \exp\left(\int (\frac{1}{y} - 2)dy\right) = \exp\left(\ln y - 2y\right) = ye^{-2y}.$$

From $(\mu\nu)' = 0$ we obtain an integrating factor μ of the given differential equation, namely: $\mu(y) = e^{2y}/y$.

Multiplying both sides of the given differential equation by $\mu(y)$, we obtain the equation

$$e^{2y}dx + (2xe^{2y} - 1/y)dy = 0.$$

From $\partial\psi/\partial x = e^{2y}$, we get $\psi(x,y) = xe^{2y} + g(y)$, which implies

$$\frac{\partial\psi}{\partial y} = 2xe^{2y} + g'(y).$$

Equating $\partial\psi/\partial y$ to $2xe^{2y} - 1/y$, we conclude that $g'(y) = -1/y$. Thus $g(y) = -\ln|y| + C_1$ for some constant C_1; in what follows we take $C_1 = 0$. Hence we obtain the family of solutions

$$xe^{2y} - \ln|y| = C,$$

where C is an arbitrary constant. Besides this family of solutions, the given differential equation has also the special solution $y = 0$.