MA 214 Calculus IV (Spring 2016)
Section 2

Homework Assignment 5
Solutions

1. Boyce and DiPrima, p. 144, Problem 8.

   Solution: The characteristic equation of the differential equation
   $$y'' - 2y' - 2y = 0$$
   is $r^2 - 2r - 2 = 0$, which by virtue of the quadratic formula has two real solutions
   $$r_1 = 1 + \sqrt{3}, \quad r_2 = 1 - \sqrt{3}.$$
   Hence the general solution of the given equation is:
   $$y = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t},$$
   where c_1 and c_2 are arbitrary constants.

2. Boyce and DiPrima, p. 144, Problem 15.

   Solution: The characteristic equation of the differential equation
   $$y'' + 8y' - 9y = 0$$
   is $r^2 + 8r - 9 = 0$, which has two real solutions $r_1 = 1$ and $r_2 = -9$. Hence the general solution of the given differential equation is:
   $$y = c_1 e^t + c_2 e^{-9t},$$
   where c_1 and c_2 are arbitrary constants. It follows from the preceding formula for the general solution that
   $$y' = c_1 e^t - 9c_2 e^{-9t}.$$
   From the initial conditions $y(1) = 1$ and $y'(1) = 0$, we obtain the simultaneous equations
   $$\begin{aligned} c_1 e + c_2 e^{-9} &= 1 \\ c_1 e - 9c_2 e^{-9} &= 0, \end{aligned}$$

whose unique solution is given by

$$c_1 = \frac{9}{10}e^{-1}, \quad c_2 = \frac{1}{10}e^{9}.$$

Hence the solution of the given initial-value problem is

$$y = \frac{9}{10}e^{t-1} + \frac{1}{10}e^{-9(t-1)}.$$

Since $y' = \frac{9}{10}\,e^{t-1}(1 - e^{-10(t-1)}) > 0$ and $y'' = \frac{9}{10}\,e^{t-1}(1 - e^{-10(t-1)}) + 9e^{-9(t-1)} > 0$ for $t > 1$, we see that $y(t)$ is increasing and is concave upward on $[1, \infty)$. Moreover, $y(t) \to \infty$ as $t \to \infty$.

3. Boyce and DiPrima, p. 144, Problem 18.

   Solution: Since $r_1 = -1/2$ and $r_2 = -2$, the characteristic equation of the differential equation in question is $(r + \frac{1}{2})(r + 2) = 0$ or $2r^2 + 5r + 2 = 0$. Hence the required differential equation is:
$$2y'' + 5y' + 2y = 0.$$

4. Boyce and DiPrima, p. 144, Problem 21.

   Solution: The characteristic equation of the given differential equation is $r^2 - r - 2 = 0$, which has two real roots $r_1 = 2$ and $r_2 = -1$. Hence the general solution is

$$y = c_1 e^{2t} + c_2 e^{-t}.$$

   From the initial conditions $y(0) = \alpha$ and $y'(0) = 2$, we obtain the simultaneous equations

$$\begin{aligned} c_1 + c_2 &= \alpha, \\ 2c_1 - c_2 &= 2, \end{aligned}$$

   from which we obtain
$$c_1 = \frac{\alpha + 2}{3}, \qquad c_2 = \frac{2(\alpha - 1)}{3}.$$
   For a given α, the solution of the given initial-value problem is

$$y(t) = \frac{\alpha + 2}{3}\,e^{2t} + \frac{2(\alpha - 1)}{3}\,e^{-t}.$$

   For the solution to approach zero as $t \to \infty$, it is necessary and sufficient that $\alpha = -2$.

5. Boyce and DiPrima, p. 164, Problem 4 and Problem 5.

   Solution: Problem 4. We have

$$e^{2-\frac{\pi}{2}i} = e^2 e^{-\frac{\pi}{2}i} = e^2\left(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right) = -ie^2.$$

   Problem 5. Note that $2^z = e^{z\ln 2}$ for any complex number z. Hence we have

$$2^{1-i} = e^{(1-i)\ln 2} = e^{\ln 2}e^{-i\ln 2} = 2(\cos(\ln 2) - i\sin(\ln 2)).$$

6. Boyce and DiPrima, p. 164, Problem 8 and Problem 10.

   Solution: Problem 8. The characteristic equation is $r^2 - 2r + 6 = 0$. Using the quadratic formula, we find the roots of the characteristic equation, which are $r_1 = 1 + i\sqrt{5}$, $r_2 = 1 - i\sqrt{5}$. Hence the general solution of the given equation is:

$$y = c_1 e^t \cos\sqrt{5}\,t + c_2 e^t \sin\sqrt{5}\,t,$$

   where c_1 and c_2 are arbitrary constants.

   Problem 10. The characteristic equation is $r^2 + 2r + 2 = 0$. Its roots are $r = -1 \pm i$. Hence the general solution of the given equation is:

$$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t,$$

   where c_1 and c_2 are arbitrary constants.

7. Boyce and DiPrima, p. 164, Problem 18.

   Solution: The characteristic equation of the given equation is $r^2 + 4r + 5 = 0$, which has conjugate complex roots $r_1 = -2 + i$, $r_2 = -2 - i$. The general solution is

$$y = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t.$$

   From the initial condition $y(0) = 1$ and $y'(0) = 0$, we get $c_1 = 1$ and the equation $-2c_1 + c_2 = 0$, respectively. It follows that $c_2 = 2$. Hence the solution of the given initial-value problem is

$$y = e^{-2t}\cos t + 2e^{-2t}\sin t.$$

   The solution describes a decaying oscillation. As $t \to \infty$, $y(t) \to 0$.

8. Boyce and DiPrima, p. 164, Problem 20.

   Solution: The roots of the characteristic equation $r^2 + 1 = 0$ are: $r = \pm i$. Hence the general solution of the given equation is:

$$y = c_1\cos t + c_2\sin t.$$

From the initial conditions $y(\pi/3) = 2$ and $y'(\pi/3) = 4$, we obtain the following two linear equations on c_1 and c_2:

$$\begin{aligned} \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 &= 2, \\ -\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 &= -4. \end{aligned}$$

The determinant of coefficients is:

$$\Delta = \begin{vmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{vmatrix} = 1.$$

It follows from Cramer's rule that

$$c_1 = \begin{vmatrix} 2 & \sqrt{3}/2 \\ -4 & 1/2 \end{vmatrix} = 1 + 2\sqrt{3}, \qquad c_2 = \begin{vmatrix} 1/2 & 2 \\ -\sqrt{3}/2 & -4 \end{vmatrix} = -2 + \sqrt{3}.$$

Hence the solution of the given initial-value problem is:

$$y = (1 + 2\sqrt{3})\cos t + (-2 + \sqrt{3})\sin t,$$

which describes a steady harmonic oscillation.

9. Boyce and DiPrima, Section 3.5, p. 172, Problem 6, and p. 173, Problem 10.

   Solution: Problem 6. The characteristic equation of the given equation is $r^2 - 6r + 9 = (r-3)^2 = 0$, which has double real root $r_1 = r_2 = 3$. The general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$.

   Problem 10. The characteristic equation of the given equation is $2r^2 + 2r + 1 = 0$, which has a pair of cojugate complex roots $r_1 = -\frac{1}{2} + \frac{1}{2}i$, $r_2 = -\frac{1}{2} - \frac{1}{2}i$. The general solution is $y = c_1 e^{-t/2}\cos(t/2) + c_2 e^{-t/2}\sin(t/2)$.

10. Boyce and DiPrima, Section 3.5, p. 173, Problem 11.

    Solution: The characteristic equation of the given equation is $9r^2 - 12r + 4 = (3r-2)^2 = 0$, which has double real root $r_1 = r_2 = 2/3$. The general solution is $y = c_1 e^{2t/3} + c_2 t e^{2t/3}$.

    From the initial conditions $y(0) = 2$ and $y'(0) = -1$, we obtain the equations

    $$c_1 = 2, \qquad \frac{2}{3}c_1 + c_2 = -1,$$

    from which we obtain $c_1 = 2$, $c_2 = -7/3$. Hence the solution of the given initial-value problem is:

    $$y = 2e^{2t/3} - \frac{7}{3}te^{2t/3} = e^{2t/3}\left(2 - \frac{7}{3}t\right).$$

    Since $e^{2t/3} > 0$, we see that $y \to -\infty$ as $t \to \infty$.