Countable Cohen-Macaulay Type and Isolated Singularities

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Question [Huneke-Leuschke]. Let R be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay type, and assume that R has an isolated singularity. Is R then necessarily of finite Cohen-Macaulay type?

Definition. A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is said to have finite (resp., countable) Cohen-Macaulay type if it has only finitely (resp., countably) many isomorphism classes of maximal Cohen-Macaulay modules.

Hypersurfaces

Examples of Finite Type. The complete ADE plane curve singularities over an algebraically closed field k are the rings $k[\![x, y, z_2, \ldots, z_r]\!]/(f)$, where f is one of the following polynomials:

$$\begin{aligned}
(A_n) &: x^{n+1} + y^2 + z_2^2 + \cdots + z_r^2,\ n \geqslant 1\\
(D_n) &: x^{n-1} + xy^2 + z_2^2 + \cdots + z_r^2,\ n \geqslant 4\\
(E_6) &: x^4 + y^3 + z_2^2 + \cdots + z_r^2\\
(E_7) &: x^3 y + y^3 + z_2^2 + \cdots + z_r^2\\
(E_8) &: x^5 + y^3 + z_2^2 + \cdots + z_r^2
\end{aligned}$$

Theorem [B-G-S]. A complete hypersurface singularity over an algebraically closed uncountable field k has (infinite) countable Cohen-Macaulay type iff it is isomorphic to one of the following

$$k[\![x, y, z_2, \ldots, z_r]\!]/(y^2 + z_2^2 + \cdots + z_r^2);$$
$$k[\![x, y, z_2, \ldots, z_r]\!]/(xy^2 + z_2^2 + \cdots + z_r^2).$$

Remark. A complete hypersurface singularity of (infinite) countable Cohen-Macaulay type has non-isolated singularities.

Dimension One, Reduced

Drozd-Roĭter conditions. If $(R, \mathfrak{m}, k)$ is a one dimensional, reduced, local, Noetherian ring with integral closure S, finitely generated as an R-module, then Wiegand et al showed R has finite Cohen-Macaulay type when the following conditions occur:

$$\dim_k(S/\mathfrak{m}S) \leqslant 3; \tag{dr1}$$

$$\dim_k\left(\frac{R + \mathfrak{m}S}{R + \mathfrak{m}^2 S}\right) \leqslant 1. \tag{dr2}$$

Theorem [Karr-Wiegand]. Let R be as above with k an uncountable field, then countable type implies finite type.

Proposition. Let x be a minimal reduction of the maximal ideal $\mathfrak{m}$. The Drozd-Roĭter conditions are equivalent to the following:

$$e(R) \leqslant 3; \tag{1}$$

$$\lambda(\overline{\mathfrak{m}^2}/x\mathfrak{m}) \leqslant 1, \tag{2}$$

Example. The following are not finite type:

- $R = k[\![t^3, t^7]\!]$; $e(R) = 3$ and $\lambda(\overline{\mathfrak{m}^2}/t^4\mathfrak{m}) = 2$.
- $R = k[\![x, y]\!]/(x^3 y - xy^3)$; $e(R) = 4$ and $\lambda(\overline{\mathfrak{m}^2}/(x + 2y)\mathfrak{m}) = 1$.

Arbitrary Dimension

Example. The following has finite type and $e(R) = n$:

$$R = k[x_1, \ldots, x_{n+1}]/\det\nolimits_2 \begin{pmatrix} x_1 & \cdots & x_n \\ x_2 & \cdots & x_{n+1} \end{pmatrix},\ n \geqslant 2$$

Definition. Let R be a homogeneous, Noetherian, Cohen-Macaulay ring with dimension d. Then we say R is Super-Stretched provided

$$\dim_k\left(\frac{R}{(x_1, \ldots, x_d)}\right)_i \leqslant 1$$

for all system of parameters $x_1, \ldots, x_d$ and for all

$$i > \sum_{j=1}^{d} \deg x_j - d + 1.$$

Examples of Super-Stretched.

- The ADE plane curve singularities, along with the two hypersurfaces of countable type, are Super-Stretched.
- The ring $R = k[\![x, y]\!]/(x^3 y - xy^3)$ is not Super-Stretched:

$$\dim_k\left(\frac{R}{(x + 2y)^2}\right)_3 = 2.$$

Main Theorem [—]. A homogeneous, Noetherian, Cohen-Macaulay ring of countable type and uncountable residue field is Super-Stretched.

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