When is an associated graded ring Cohen-Macaulay if it is a domain?

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What is an associated graded ring?

Let (R, m) be a Noetherian local ring with maximal ideal m. Let I be an R-ideal. There are graded algebras associated to the ideal I. Here we review the blowup algebras of I and their relationships,

$$R \subset R[It] \subset R[It, t^{-1}] \subset R[t, t^{-1}].$$

The rings $R[It]$ and $R[It, t^{-1}]$ are called Rees algebra and extended Rees algebra, respectively. The associated graded ring with respect to I, $gr_I(R)$, is

$$gr_I(R) := R/I \oplus I/I^2 \oplus I^2/I^3 \cdots \cong R[It, t^{-1}]/(t^{-1}) \cong R[It]/IR[It].$$

Since t^{-1} is a homogeneous non zerodivisor on $R[It, t^{-1}]$, it follows, for instance, that $gr_I(R)$ is Cohen-Macaulay (Gorenstein) if and only if $R[It, t^{-1}]$ is Cohen-Macaulay (Gorenstein), respectively.

Remark

Most ring-theoretic properties (e.g. reduced, irreducible, integrally closed, Cohen-Macaulay, etc) of the associated graded ring pass to the ambient ring R.

When is an associated ring Cohen-Macaulay?

A great deal of research has been done on this question. One approach uses numerical criteria based on analytic spread, reduction number, or Hilbert-Samuel multiplicity. [Ulrich] summarizes such results. Another approach is assuming the associated graded ring is a domain. More results are known when an associated graded ring becomes Gorenstein. We list one here.

[Hochster, HSV]

Let (R, m) be a Gorenstein local ring, and let I be an ideal of R whose associated graded ring $gr_I(R)$ is a Cohen-Macaulay domain. Then $gr_I(R)$ is also Gorenstein.

Question 1

Can we remove the assumption "the associated graded ring is Cohen-Macaulay" in the statement above? To this end Heinzer, Kim and Ulrich raised the following question.

[HKU, Question 4.11]

Let (R, m) be a Gorenstein local ring and I an m-primary ideal. Is the extended Rees algebra $R[It, t^{-1}]$ Gorenstein if it is quasi-Gorenstein?

Definition: A ring is called quasi-Gorenstein if it is isomorphic to its canonical module (but not necessarily Cohen-Macaulay).

Remarks on the question

- [HKU, Corollary 4.12] Let (R, m) be a 2-dimensional regular local ring and let I be an m-primary ideal. If $R[It, t^{-1}]$ is quasi-Gorenstein, then $R[It, t^{-1}]$ is Gorenstein.
- [HH, Example 4.7] shows that the assumption $\sqrt{I} = m$ cannot be removed.

Special case of [HKU, Question 4.11]

Let (R, m) be a Gorenstein local ring. Is $gr_m(R)$ Gorenstein if $gr_m(R)$ is a domain?

Remark: If $gr_m(R)$ is a domain then $R[It, t^{-1}]$ is quasi-Gorenstein.

- More generally, one can ask.

Question 2

Let (R, m) be a Cohen-Macaulay local ring. Is $gr_m(R)$ Cohen-Macaulay if it is a domain?

Positive answer when ecodim is small

The embedding codimension measures how far a ring is from a regular local ring. That is, ecodim$(R) = h$ means that $\hat{R} \cong S/I$ where (S, n) is a regular local ring and $I \subset n^2$ is a height h ideal in S.

The question has a positive answer if

- ecodim$(R) = 1$
- ecodim$(R) = 2$ and R is a complete intersection ring.

How much can we relax the condition "complete intersection" if ecodim$(R) = 2$? To this end we ask the following question.

Question 3

Let (R, m) be a Cohen-Macaulay local ring. Assume that ecodim$(R) = 2$ and R is an almost complete intersection. Is $gr_m(R)$ Cohen-Macaulay if it is a domain?

Theorem

Let (R, m) be a Cohen-Macaulay local ring with ecodim$(R) = 2$, i.e. $\hat{R}$ can be written as S/I where (S, n) is a regular local ring and I is a height two Cohen-Macaulay ideal. Assume that I is 3-generated and ord$_n(I)$ is at most 4. If $gr_m(R)$ is a domain then it is Cohen-Macaulay.

Definition: ord$_n(I) = \max\{i | I \subset n^i\}$

Sketch of the proof

- By [HB], one can write $I = I_2(M)$ where
$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}.$$
- Modify M to obtain
$$M^* = \begin{pmatrix} x_{11}^* & x_{12}^* & x_{13}^* \\ x_{21}^* & x_{22}^* & x_{23}^* \end{pmatrix}$$
with grade $J = 2$ for $J := I_2(M^*)$. Here x^* denotes the leading form of an element $x \in S$ in $gr_n(S)$.
- Then we can lift the relations among the generators of J to R, which is equivalent to say $J = I^*$ where I^* denotes the leading ideal of I, i.e. the ideal generated by the leading forms of all elements of I.
- This shows that $gr_m(R) \cong gr_n(S)/I^*$ is Cohen-Macaulay by [HB].

Remark: the condition "domain" is necessary

Let $S = \mathbb{C}[[x, y, u, v, w]]$, $n = (x, y, u, v, w)$ and $I = I_2(M)$ where

$$M = \begin{pmatrix} x^2 + u^3 & 0 & xv + u^3 \\ xy + u^3 & xw + x^3 & 0 \end{pmatrix}.$$

Set $R = S/I$. Then $gr_m(R)$ is not Cohen-Macaulay and it is not a domain, although the other assumptions of the theorem are satisfied.

References

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