Abstract

We establish a formula for the Chern number in a d-dimensional local ring involving modified Koszul complex and specifically give the formula in dimension one and two. As applications we unify several results for Chern number in local rings of dimension \( \leq 2 \).

Introduction

Let \((R, m)\) be a local ring and \(I\) be an \(m\)-primary ideal. Then a sequence of ideals \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) is called an \(I\)-admissible filtration if there exists \(k \in \mathbb{N}\) such that for all \(n, m \in \mathbb{Z}\) the following three conditions hold

- (a) \(I_{n+1} \subseteq I_n\)
- (b) \(I_n + m \subseteq I_{n+1}\)
- (c) \(I^n \subseteq I \subseteq I^{n+1}\) for all \(n \geq k\).

Marley showed that if \(I\) is an \(I\)-admissible filtration then the function \(H_n(I) = \lambda(R/I_n)\) where \(\lambda\) denotes length as \(R\)-module coincides with the Hilbert polynomial associated with the \(I\)-admissible filtration \(I\). The integers \(e(I)\) for \(1, \ldots, d\) are called the Hilbert coefficients of \(I\). The coefficient \(e(I)\) is called the Chern number of \(I\).

Formula for the Chern number involving modified Koszul complex

Let \((R, m)\) be a \(d\)-dimensional local ring and \(I_{n} \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Let \(y_1, \ldots, y_d \in I\). Define for \(n \in \mathbb{Z}\), the modified Koszul complex \(C(n, I)\)

\[
0 \longrightarrow R/I_n \longrightarrow (R/I_{n+1})^d \longrightarrow \cdots \longrightarrow (R/I_{d-1})^{(d-1)} \longrightarrow R/I_d \longrightarrow 0
\]

(1)

with \(I_n = R\) for \(n = 0\) and the differentials are induced by the ideals of the Koszul complex \(K = \langle x_1, \ldots, x_d, R \rangle\). In other words we have an exact sequence of complexes

\[
0 \longrightarrow Kn^{(I)}(x_1, \ldots, x_d) \longrightarrow C(n, I) \longrightarrow 0
\]

(2)

where \(K^{(I)}(x_1, \ldots, x_d) = \langle x_1, \ldots, x_d, R \rangle\):

\[
0 \longrightarrow I_n \longrightarrow \cdots \longrightarrow (I_{n+1})^{(d-1)} \longrightarrow I_{d-1} \longrightarrow 0.
\]

The Euler characteristic of \(C(n, I)\) is defined as

\[
\chi(C(n, I)) = \sum_{i=0}^{d} (-1)^i \lambda(H_i(C(n, I))).
\]

Then from the exact sequence (2) we have

\[
\chi(K) = \chi(K^{(I)}(I)) + \chi(C(n, I)).
\]

Main Theorem. Let \((R, m)\) be a \(d\)-dimensional local ring. Let \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Then

\[
e(I) = \sum_{n=1}^{\infty} \lambda(I_n/I).
\]

Sketch of Proof. Using the theory of numerical functions we observe that

\[
e(I) = \sum_{n=1}^{\infty} \Delta^d P_n(x) - H_t(n).
\]

By Serre’s Theorem we have \(e(I) = \chi(K)\). Hence from the exact sequence (2) we have

\[
\Delta^d P_n(x) - H_t(n) = \chi(K^{(I)}(I))
\]

Consequences of Main Theorem

Corollary 1. Let \((R, m)\) be a 1-dimensional local ring and \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Let \(J = (x) \subseteq I\) be a minimal reduction. Then

(a) \(e(I) = \sum_{n=1}^{\infty} \lambda(I_n/I_{n+1}) - \lambda((0 : x) \cap I_n/I_{n+1})\).

(b) If \(R\) is Cohen-Macaulay then

\[
e(I) = \sum_{n=1}^{\infty} \lambda(I_n/I_{n+1}).
\]

Corollary 2. Let \((R, m)\) be a 2-dimensional local ring. Let \(J = (x, y)\) be a parameter ideal. Let \(J' = (I)\) be a \(J\)-admissible filtration and \(depth G(J) \geq 1\) and \(x' \in J/J'\) be a nonzerodivisor in \(G(J')\). Then

\[
e(J) = e(J) - \lambda(R/J) + \sum_{n=2}^{\infty} \lambda \left( \frac{I_n/J_{n-1}}{J_{n-1}} \right) - \lambda \left( \frac{(x') \cap J_{n-1}}{(x') J_{n-1}} \right)
\]

Corollary 3. Let \((R, m)\) be a 2-dimensional analytically unramified local ring and \(J = (x, y)\) be a parameter ideal. Then

\[
\pi_t(J) = e(J) - \lambda(R/J) + \sum_{n=2}^{\infty} \lambda \left( \frac{I_n/J_{n-1}}{J_{n-1}} \right) - \lambda \left( \frac{(x') \cap J_{n-1}}{(x') J_{n-1}} \right).
\]

Applications

Rees, 1961

Let \((R, m)\) be a 1-dimensional local ring. Let \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Let \(J \subseteq I\) be a parameter ideal such that \(e(J) = e(I)\). Then \(J\) is a reduction of \(I\).

Lipman, 1971

Let \((R, m)\) be a 1-dimensional Cohen-Macaulay ring and \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Let \(J = (x)\) be a minimal reduction of \(I\). Then for all \(n \geq 1\)

\[
\lambda(I_{n-1}/I) \leq e(I)
\]

and

\[
\lambda(I_{n-1}/I) = e(I)\]

Huneke, 1987

Let \((R, m)\) be a 1-dimensional Cohen-Macaulay ring and \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Assume that

\[
e(I) = e(I) - \lambda(R/I).
\]

Then

\[
H_n(I) = P_n(x)\]

for all \(n \geq 1\).

If \((x)\) is a minimal reduction of \(I\) then \(I_n = I_{n-1}\) for all \(n \geq 2\).

Huneke, 1987

Let \((R, m)\) be a 2-dimensional Cohen-Macaulay local ring and \(I\) be an \(m\)-primary ideal and \(I = I_n \subseteq I_{n+1} \subseteq \cdots \subseteq I\) be an \(I\)-admissible filtration. Let \(J = (x, y)\) be a minimal reduction of \(I\). Then for all \(n \geq 2\)

\[
\Delta^d P_n(x) - H_t(n) = \lambda \left( \frac{I_n/J_{n-1}}{J_{n-1}} \right) - \lambda \left( \frac{(x') \cap J_{n-1}}{(x') J_{n-1}} \right).
\]

Sally, 1992

Let \((R, m)\) be a 1-dimensional Cohen-Macaulay ring and \(I\) be an \(m\)-primary ideal. Assume that

\[
e(I) = e(I) - \lambda(R/I) + 1.
\]

Then

\[
H_n(I) = P_n(x)\]

for all \(n \geq 1\).

If \((x)\) is a minimal reduction of \(I\) then \(I^2 = I^2\).

References

4. J. D. Sally, Hilbert coefficients and reduction number 2, J. Algebraic Geom 1 (1992), 325-333.