33 Zeros of Polynomials

The zeros of a polynomial $f(x)$ are the solutions of the equation

$$ f(x) = 0 $$

Observe that each "real" zero corresponds to the $x$-intercept of the graph of $f$.

* Fundamental Theorem of Algebra

(by Friedrich Gauss, 1777-1855)

If $f(x)$ is a polynomial of positive degree and complex coefficients, then $f(x)$ has at least one complex zero.

I.e.

$$ f(x) = (x - c_1) g(x) $$

Thus, at least in theory we conclude that the above $f$ can be factored as

$$ f(x) = a (x - c_1)(x - c_2) \cdots (x - c_n) $$

Each $c_k \in \mathbb{C}$ is a zero of $f(x)$. 
* Notice that a polynomial of degree $n$ has at most $n$ different zeros.

Indeed, some zeros could be repeated.

**Ex:** \[ f(x) = x^3 + x^2 - 5x + 3 \]

\[ = (x+3)(x-1)^2 \]

$x = 1$ is a zero of multiplicity 2
$x = -3$ is a simple zero

The graph of $f(x)$ looks like:

In general, if there is a factor $(x-c)^m$ in $f(x)$ we say that $c$ is a zero of multiplicity $m$.
Descartes' Rule of Signs

Let \( f(x) \) be a polynomial with real coefficients and a non-zero constant term:

1. The number of positive real zeros of \( f(x) \) either is equal to the number of variations of sign in \( f(x) \) or is less than that number by an even number.

2. The number of negative real zeros of \( f(x) \) is either equal to the number of variations of sign in \( f(-x) \) or is less than that number by an even number.

Ex: \( f(x) = 2x^5 - 7x^4 + 3x^2 + 6x - 5 \)

\[ \begin{array}{c}
+ & - & - & + & - \\
3 \text{ variations}
\end{array} \]

\[ f(-x) = -2x^5 - 7x^4 + 3x^2 - 6x - 5 \]

\[ \begin{array}{c}
- & + & + & - & - \\
2 \text{ variations}
\end{array} \]

Possible Cases:

<table>
<thead>
<tr>
<th># of positive real zeros</th>
<th>3</th>
<th>3</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td># of negative real zeros</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td># of imaginary complex zeros</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>TOTAL # of zeros</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
Note: the constant term has to be positive.

Otherwise, write \( f(x) \) as

\[
f(x) = x^n \cdot g(x)
\]

where \( g(x) \) has to be a constant term.

We would like now to find explicit upper and lower bounds for the real zeros of a polynomial \( f(x) \):

![Real zeros of \( f(x) \)]

**Theorem 1**

Suppose \( f(x) \) has real coefficients and positive leading term. Suppose that \( f(x) \) is synthetically divided by \( x-c \).

1. If \( c > 0 \) and if all \( \#'s \) in the third row of the division process are either positive or zero, then \( c \) is an upper bound for the real zeros of \( f \).

2. If \( c < 0 \) and if all \( \#'s \) in the third row are alternately positive or negative, then \( c \) is a lower bound for the real zeros of \( f \).
Example:

Let \( f(x) = 2x^3 + 5x^2 - 8x - 7 \)

Consider Descartes' rule of signs:

\[
f(x) = 2x^3 + 5x^2 - 8x - 7
\]

\[
f(-x) = -2x^3 + 5x^2 + 8x - 7
\]

Thus \( f(x) \) has 1 positive real root, and either 2 or 0 negative roots.

\[
\begin{array}{cccc|c}
 1 & 2 & 5 & -8 & -7 \\
 2 & 7 & -1 & -8 \\
 2 & 9 & 10 & 13 \\
\end{array}
\]

Thus \( c = 2 \) is an upper bound for the positive real zeros of \( f \).

\[
f(x) = (x-2)(x^2 + 9x + 10) + 13 \]

i.e., for values greater than or equal to 2 \( f(x) \) is always positive.

\[
\begin{array}{cccc|c}
 2 & 5 & -8 & -7 \\
 & -8 & +12 & -16 \\
 & 2 & -3 & 4 & -23 \\
\end{array}
\]

\[
f(x) = (x+4)(2x^2 - 3x + 4) - 23 \]

i.e., for all values smaller than or equal to -4 \( f(x) \) is always negative!
Theorem 2

Suppose \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \)

All zeros of \( f \) are in the symmetric interval \((-M, M)\)

where \( M = \max \left( \frac{|a_n|, |a_{n-1}|, \ldots, |a_1|, |a_0|}{} \right) + 1 \)

\[ M = \frac{\max(2, 5, 8, 7)}{2} + 1 = \frac{8}{2} + 1 = 5 \]

Thus the zeros of \( f \) are inside \((-5, 5)\)

Notice that this result is easy to verify in practice. That's why the bound is less sharp!!