4.3 The Natural Exponential Function

We discussed — in the previous lecture — that the amount of money deposited in a bank grows according to the formula

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}$$

where

- $n$ = # of interest periods per year
- $r$ = interest rate
- $t$ = time in years

If $n$ grows, of course, we expect better profit.

**Example:** $P_0 = 1,000$, $r = 9\%$ or 0.09

- If $n = 4$ (i.e. the interest is computed each quarter)
  $$P(1) = 1,000 \left(1 + \frac{0.09}{4}\right)^4 = 1,093.08$$

- If $n = 12$ (i.e. the interest is computed each month)
  $$P(1) = 1,000 \left(1 + \frac{0.09}{12}\right)^{12} = 1,093.81$$

- If $n = 52$ (i.e. the interest is computed each week)
  $$P(1) = 1,000 \left(1 + \frac{0.09}{52}\right)^{52} = 1,094.09$$

- If $n = 365$ (i.e. the interest is computed each day)
  $$P(1) = 1,000 \left(1 + \frac{0.09}{365}\right)^{365} = 1,094.16$$

- If $n = 8760$ (i.e. the interest is computed each hour)
  $$P(1) = 1,000 \left(1 + \frac{0.09}{8760}\right)^{8760} = 1,094.17$$
* If \( n = 525,600 \) (i.e. the interest is computed each minute),
\[
P'(1) = \frac{1,000}{525,600} \left( 1 + \frac{0.09}{525,600} \right)^{525,600} = \$ 1,094.17
\]

Thus \( P'(1) \) approaches a fixed value as \( n \) increases.

If \( n \to \infty \) we say that the interest is compounded continuously.

What's the formula in this case?
\[
P_o (1 + \frac{r}{n})^{nt} = P_o \left[ (1 + \frac{1}{n})^{\frac{n}{r}} \right]^{rt} \quad \text{as} \quad n \to \infty
\]

This is because the sequence of numbers
\[
(1 + \frac{1}{q})^q \quad \text{approaches} \quad e = 2.71828 \quad \text{as} \quad q \to \infty
\]

as you can convince yourself after building a chart of values:

<table>
<thead>
<tr>
<th>( q )</th>
<th>( q = 1 )</th>
<th>( q = 10 )</th>
<th>( q = 1,000 )</th>
<th>( q = 1,000,000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1 + \frac{1}{q})^q )</td>
<td>2</td>
<td>2.59374</td>
<td>2.71692</td>
<td>2.71828</td>
</tr>
</tbody>
</table>

Thus:
\[
P(t) = P_o e^{rt}
\] is the formula in the interest compounded continuously.
The natural exponential function is defined by \( f(x) = e^x \) and \( \frac{d}{dx} e^x = e^x \).

Its graph looks like:

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**Ex:** How much money invested at the interest rate of \( r = 9.5\% \) compounded continuously will amount to \( \$15,000 \) after 4 years?

\[
15,000 = e^{0.095 \cdot 4}
\]

\[
\therefore \quad \frac{15,000}{e^{0.38}} = 15,000 \cdot e^{-0.38}
\]

\[
= \$10,257.92
\]

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**Ex:** An investment of \( \$400 \) increased to \( P = \$890.20 \) in 16 years. Find the interest rate \( r \) if the interest was compounded continuously.

We need to solve the equation

\[
890.20 = 400 \cdot e^{4r}
\]

\[
\Leftrightarrow e^{4r} = \frac{890.20}{400} = 2.2255
\]

We can check with the calculator that \( r = 0.05 \) (or 5\%) works.

The exact answer is:

\[
r = \frac{1}{4} \ln(2.2255)
\]

(we'll see this later!)
Ex. The function \( f(x) = \frac{e^x + e^{-x}}{2} \)

is called the hyperbolic cosine function. Observe that \( f'(x) = \frac{e^{-x} + e^{-(x)}}{2} = f(x) \)

i.e. \( f(x) \) is even.

Its graph looks like:

\[ \cosh(x) \]

It can be shown that a uniform flexible cable hangs from 2 poles of the same height according to a shape described by:

\[ g(x) = \frac{a}{2} \left( e^{x/a} + e^{-x/a} \right) \]

The case \( a=1 \) means:

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]

Similarly, the hyperbolic sine is defined by:

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

This function is odd!

Observe that:

\[
\frac{[\cosh(x)]^2 - [\sinh(x)]^2}{\left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2} = \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} = 1
\]

Ex.: Find the zeros of

\[ -x^2 e^{-x} + 2x e^{-x} = 0 \]

\[ e^{-x} [2x - x^2] = 0 \quad \Rightarrow \quad 2x - x^2 = 0 \quad \Rightarrow \quad x = 0, 2 \quad \text{as} \ e^x \neq 0 \]