4.4 Logarithmic functions

We defined $f(x) = a^x$ for $0 < a < 1$ or $a > 1$.

We also noticed that these functions are one-one, hence they are invertible.

The inverse of $y = a^x$ is called the "<u>logarithm</u> of <u>x with base a</u>"

and denoted by $y = \log_a x$ ‖‖

Since these 2 functions are inverses of each other, we have that

$$a^{\log_a x} = x \quad \underline{\text{and}} \quad \log_a(a^x) = x$$

The first identity says that if we call $y = \log_a x$ then $a^y = x$

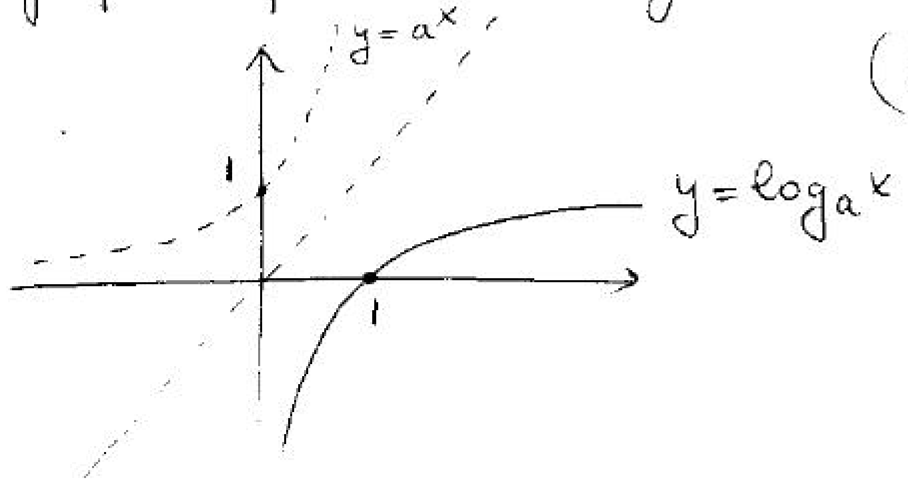
i.e. the logarithm in base a of x is that number y to which the base a must be raised in order to get back x.

<u>Ex</u>:
- $\log_5 u = 2 \longleftrightarrow 5^2 = u \rightsquigarrow u = 25$
- $\log_b 8 = 3 \longleftrightarrow b^3 = 8 \longrightarrow b = 2$

* $\log_5 1 = y \iff 5^y = 1 \rightsquigarrow y = 0$
* $K = H - Ca^t \iff -K + H = Ca^t \iff t = \log_a\left(\frac{H-K}{C}\right)$

* Since $y = \log_a x$ is the inverse of $y = a^x$ the graph of the logarithm is :

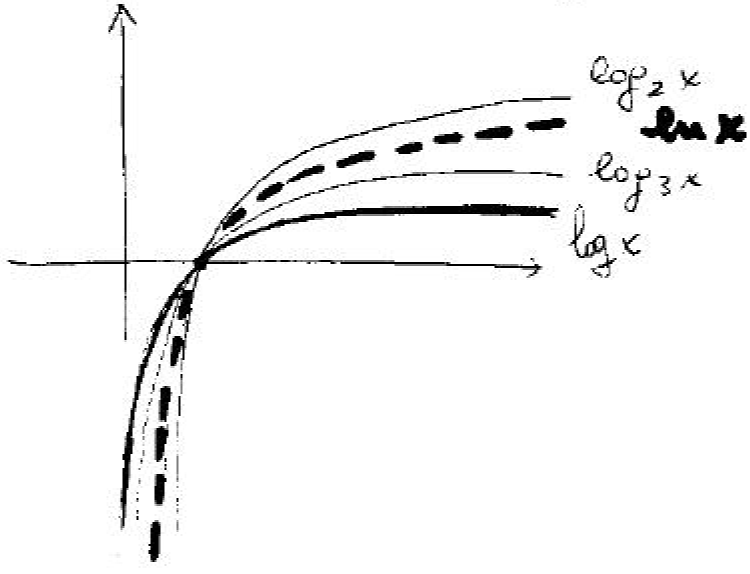
(here $a > 1$)



* Special cases of interest are when $a = e, 10$

$\log_e x$ is denoted $\ln x$ (LN on calc)

$\log_{10} x$ is denoted $\log x$ (LOG on calc)



* Clearly, also $y = \log_a x$ is a one-to one function so that

$$x_1 \neq x_2 \implies \log_a x_1 \neq \log_a x_2$$

OR

$$\log_a x_1 = \log_a x_2 \implies x_1 = x_2$$

* Solve the equations:

$\log_3(x+4) = \log_3(1-x)$

$\longrightarrow x+4 = 1-x \longrightarrow 2x = -3 \longrightarrow x = -3/2$

Notice that when we plug in $-3/2$ in both sides the argument of $\log_3$ is positive

$\log_7(x-5) = \log_7(6x) \longrightarrow x-5 = 6x$

$\longrightarrow 5x = -5 \longrightarrow x = -1$ BUT

when we substitute we get $\log_7(-6) = \log_7(-6)$ which doesn't make sense. So it has no solution!!

* $e^{x \ln 3} = 27$

How do we solve this? <u>Useful fact</u>

Set $r = \log_a u \iff u = a^r$

$\iff$ for any c $\quad u^c = (a^r)^c = a^{rc} = a^{cr}$

$\iff \log_a u^c = \log_a a^{cr} = c\boxed{r} = c \log_a u$

$$\therefore \boxed{\log_a u^c = c \log_a u}$$

In our example $\quad \ln e^{x \ln 3} = \ln 27$

$$x \ln 3 = \ln 3^3 \underset{\uparrow}{=} 3 \ln 3$$

$$\therefore x = 3$$

* If we start with q_0 milligrams of radium the amount q remaining after t years is given by the formula

$$q = q_0\, 2^{-t/1,600}$$

Express t in terms of q and q_0.

<u>Ans</u>: $\quad q/q_0 = 2^{-t/1,600} \rightsquigarrow \log_2\left(\frac{q}{q_0}\right) = \log_2 2^{-t/1,600}$

$\rightsquigarrow \log_2\left(\frac{q}{q_0}\right) = -\frac{t}{1600} \rightsquigarrow \boxed{t = -1,600 \log_2\left(\frac{q}{q_0}\right)}$