7.5 Properties of Logarithms

Let u and w be positive real numbers
THEN we have the following properties

* $\log_a(uw) = \log_a(u) + \log_a(w)$
* $\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(w)$
* $\log_a(u^c) = c \log_a u$ $\quad$ c any number

Why are they true?

Write $r = \log_a u \iff a^r = u$ $\qquad$ $s = \log_a w \iff a^s = w$

Now: $a^{r+s} = a^r \cdot a^s = uw \iff \log_a(uw) = r + s = \log_a u + \log_a w$

Similarly,

$a^{r-s} = a^r \cdot \frac{1}{a^s} = \frac{u}{w} \iff \log_a\left(\frac{u}{w}\right) = r - s = \log_a u - \log_a w$

Finally: $u^c = (a^r)^c = a^{rc} \iff \log_a u^c = rc = c \log_a u$

NOT TRUE:
$$\begin{cases} \log_a(u+w) \neq \log_a(u) + \log_a(w) \\ \log_a(u-w) \neq \log_a(u) - \log_a(w) \end{cases}$$

* Ex: Pareto's law for capitalist countries states that the relationship between annual income x and the number y of individuals whose income exceeds x is

$$\log y = \log b - k \log x$$

where b and k are positive constants. Solve this equation for y

Ans: $\log y = \log\left(\frac{b}{x^k}\right)$

Thus $y = b/x^k$

Notice: that y and x^k are inversely proportional (if x is increased then y decreases and vice versa)

* Express as one logarithm

$$\log_4(x) - \log_4(7y) = \log_4\left(\frac{x}{7y}\right)$$

* Express as one logarithm

$$5 \log_a x - \frac{1}{2} \log_a(3x-4) - 3 \log_a(5x+1)$$

$$= \log_a x^5 - \log_a (3x-4)^{1/2} - \log_a (5x+1)^3$$

$$= \log_a (x^5) - \left[\log_a \sqrt{3x-4} + \log_a (5x+1)^3 \right]$$

$$= \log_a (x^5) - \log_a \left[(5x+1)^3 \sqrt{3x-4} \right]$$

$$= \log_a \left[\frac{x^5}{(5x+1)^3 \sqrt{3x-4}} \right]$$

* Express as one logarithm

$$2 \ln x - 4 \ln\left(\frac{1}{y}\right) - 3 \ln (xy) =$$

$$= \ln x^2 - \ln\left(\frac{1}{y^4}\right) - \ln (x^3 y^3) =$$

$$= \ln x^2 - \left[\ln\left(\frac{1}{y^4}\right) + \ln (x^3 y^3) \right] =$$

$$= \ln x^2 - \left[\ln \left(\frac{x^3 y^3}{y^4} \right) \right] = \ln x^2 - \ln\left(\frac{x^3}{y}\right)$$

$$= \ln \left[\frac{x^2}{\frac{x^3}{y}} \right] = \ln \left[\frac{y}{x} \right]$$

* Solve the equation

$$\log (x+2) - \log x = 2 \log 4$$

$$\log \left(\frac{x+2}{x} \right) = \log 4^2$$

$$\Longleftrightarrow \quad \frac{x+2}{x} = 16 \rightsquigarrow x+2 = 16x \rightsquigarrow \boxed{x = \frac{2}{15}}$$

* Solve the equation

$\log_6(x+5) + \log_6(x) = 2$

$\log_6[x(x+5)] = 2 \cdot 1 = 2 \cdot \log_6 6$ ← NOTICE

$\log_6[x^2+5x] = \log_6 6^2 \iff x^2+5x = 36$

$x^2+5x-36=0 \qquad (x+9)(x-4)=0$

~~$x=-9$~~ $\quad x = 4$

if you plug it inside the origin equation we get
$\log_6(-4) + \log_6(-9) = 2$

* Solve the equation

$\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$

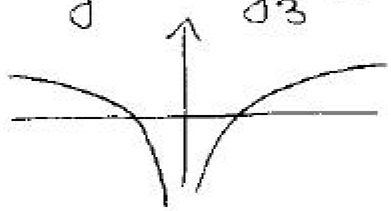
$\log_2\left(\frac{x+3}{x-3}\right) = \log_3 3^2 + 4^{\log_4 3} = 2+3$

$\log_2\left(\frac{x+3}{x-3}\right) = 5 \iff \log_2\left(\frac{x+3}{x-3}\right) = \log_2 2^5$

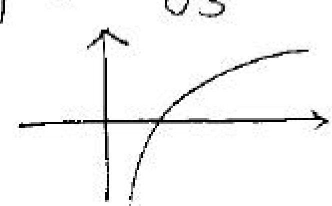
$\frac{x+3}{x-3} = 32 \rightsquigarrow x+3 = 32x-96 \rightsquigarrow x = \frac{99}{31}$

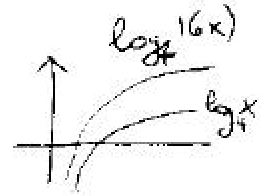
* Sketch a graph:

$y = \log_3 x^2$



$y = 2\log_3 x$





* $y = \log_4(16x) = \log_4 16 + \log_4 x = 2 + \log_4 x$