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1.3 Algebraic Expressions

* We begin with a bunch of variables {x, y, z, etc...} and real numbers. An algebraic expression looks like

$$\frac{2xy + \frac{3}{x^2}}{\sqrt[3]{y-1}}$$

* its value is what you get when replacing x and y with some numbers

* its domain is all the possible choices of x and y that makes sense for the expression

in our case $x \neq 0$, $y \neq 1$

Eg: domain of $x^3 - 5x + \frac{6}{\sqrt{x}}$ is $x > 0$

* Typical algebraic expressions are polynomials and rational expressions

$$3x^4 + 5x^2 - 7x + 4 \qquad\qquad \frac{6x^2 - 5x + 4}{x^2 - 9}$$

leading coefficient

4 = degree of polynomial

We need to be able to

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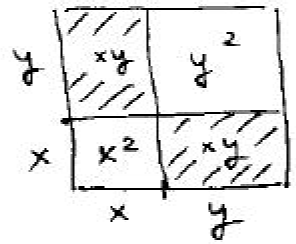
* Multiplying polynomials



eg. $(x+y)^2 = x^2 + 2xy + y^2$

$(x-y)^2 = x^2 - 2xy + y^2$

$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$

in general the coeff for $(x \pm y)^n$ comes from Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \quad \text{etc.}$$

$(x-y)(x+y)$ is also important

$= x^2 - y^2$

* $(3u-1)(u+2) + 7u(u+1) =$

$= 3u^2 + 6u - u - 2 + 7u^2 + 7u =$

$= 10u^2 + 12u - 2$

* $(\sqrt{x} + \sqrt{y})^2 (\sqrt{x} - \sqrt{y})^2 =$

$= \left[(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})\right]^2 = \left[(\sqrt{x})^2 - (\sqrt{y})^2\right]^2 =$

$= (x-y)^2 = x^2 - 2xy + y^2$

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* Factoring polynomials

- $7x^2+10x-8 = (7x \;\_\;)(x \;\_\;)$

  if possible

  factors of -8 e.g. "-4" and "2"

  $= (7x-4)(x+2)$

- $\underbrace{5x^3+10x^2}\underbrace{-20x-40}$

  $5x^2(x+2) - 20(x+2) = (x+2)[5x^2-20] =$

  $= 5(x+2)[x^2-4) = 5(x+2)(x+2)(x-2) =$

  $= 5(x+2)^2(x-2)$

* Simplify rational expressions

- $\frac{5x}{2x+3} - \frac{6}{2x^2+3x} + \frac{2}{x}$

  $x(2x+3)$

  $= \frac{(5x)x - 6 + 2(2x+3)}{x(2x+3)} = \frac{5x^2 \cancel{-6} + 4x \cancel{+6}}{x(2x+3)}$

  $= \frac{x(5x+4)}{x(2x+3)} = \frac{5x+4}{2x+3}$

- $\frac{2x+6}{x^2+6x+9} + \frac{5x}{x^2-9} + \frac{7}{x-3} =$

  $(x+3)^2$ $(x+3)(x-3)$

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$$= \frac{(2x+6)(x-3) + 5x(x+3) + 7(x+3)^2}{(x+3)^2(x-3)} =$$

$$= \frac{2x^2 - \cancel{6x} + \cancel{6x} - 18 + 5x^2 + 15x + 7x^2 + 42x + 63}{(x+3)^2(x-3)}$$

$$= \frac{14x^2 + 57x + 45}{(x+3)^2(x-3)} = \frac{(14x+15)(x+3)}{(x+3)^2(x-3)}$$

$$= \frac{14x+15}{(x+3)(x-3)} = \frac{14x+15}{(x^2-9)}$$

- $$\frac{y^{-2} - x^{-2}}{y^{-2} + x^{-2}} = \frac{\frac{1}{y^2} - \frac{1}{x^2}}{\frac{1}{y^2} + \frac{1}{x^2}} =$$

$$= \frac{\frac{x^2-y^2}{x^2y^2}}{\frac{x^2+y^2}{x^2y^2}} = \frac{x^2-y^2}{\cancel{x^2y^2}} \cdot \frac{\cancel{x^2y^2}}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

- $$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} =$$

not defined when $h=0$

$$= \frac{\cancel{x} - \cancel{x} - h}{x(x+h)} \cdot \frac{1}{h} = \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-1}{x(x+h)}$$

defined when $h=0$

(13) * Rationalizing a numerator

$$\cdot \ \frac{\sqrt{a}-\sqrt{b}}{a^2-b^2} = \text{when } a=b = \frac{0}{0}$$

$$= \frac{\sqrt{a}-\sqrt{b}}{a^2-b^2} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{(\sqrt{a})^2-(\sqrt{b})^2}{(a^2-b^2)(\sqrt{a}+\sqrt{b})}$$

$$= \frac{\cancel{a-b}}{(a+b)\cancel{(a-b)}(\sqrt{a}+\sqrt{b})} = \frac{1}{(a+b)(\sqrt{a}+\sqrt{b})}$$

* Simplifying a fractional expression

$$\frac{(x^2+4)^{1/3}\cdot(3) - (\cancel{3}x)\left(\frac{1}{\cancel{3}}\right)(x^2+4)^{-2/3}(2x)}{\left((x^2+4)^{1/3}\right)^2}$$

$$= \frac{3(x^2+4)^{1/3} - 2x^2\cdot\frac{1}{(x^2+4)^{2/3}}}{(x^2+4)^{2/3}}$$

$$= \frac{\frac{3(x^2+4)-2x^2}{(x^2+4)^{2/3}}}{(x^2+4)^{2/3}} = \frac{3x^2+12-2x^2}{(x^2+4)^{4/3}}$$

$$= \frac{x^2+12}{(x^2+4)^{4/3}}$$

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Other useful things to remember

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

e.g. $x^6 - 27y^3 =$

$$= (x^2)^3 - (3y)^3 =$$

$$= (x^2 - 3y)\left[x^4 + 3x^2y + 9y^2\right]$$

e.g. $a^6 - b^6 =$

$$= (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3) =$$

$$= (a+b)(a^2 - ab + b^2)(a-b)(a^2 + ab + b^2)$$

$$= (a+b)(a-b)(a^2 - ab + b^2)(a^2 + ab + b^2)$$