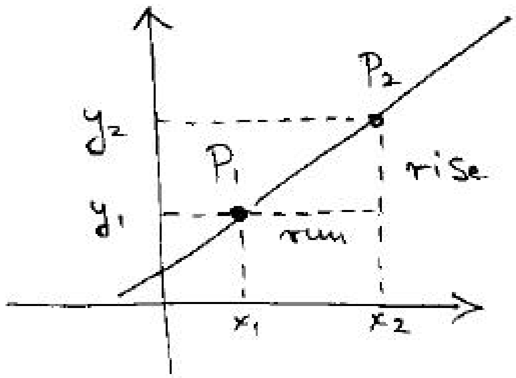
2.3 Lines

* Given 2 points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ then the graph of the line through P_1 and P_2 looks like :



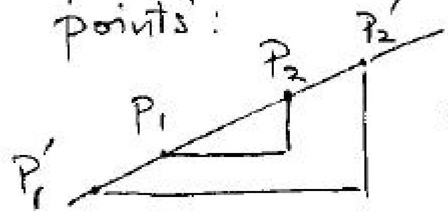
How do we describe the line with an algebraic equation?

We first define the <u>slope</u> :

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

notice

* It is a property of similar triangles that if we pick different points P_1' and P_2' the slope of the line doesn't depend on the points :



the ratio is the same !!

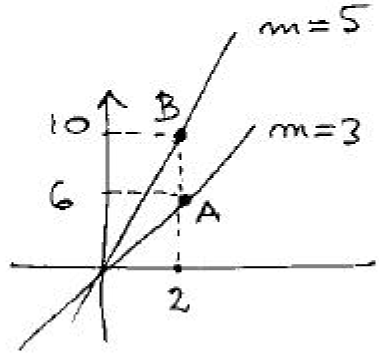
* Write the slope of the line through

$O(0,0)$ and $A(2,6)$

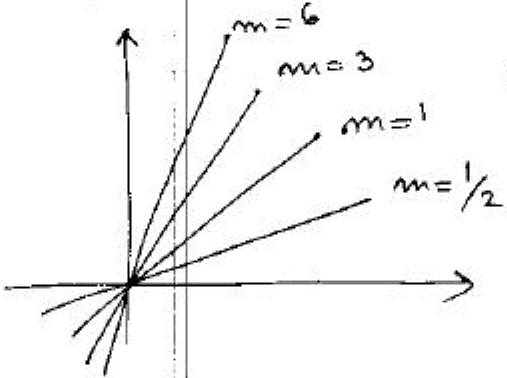
$\rightsquigarrow m = \frac{6-0}{2-0} = \frac{6}{2} = 3$

$O(0,0)$ and $B(2,10)$

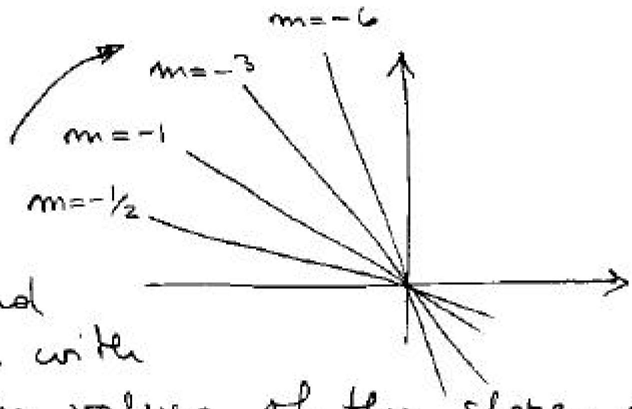
$\rightsquigarrow m = \frac{10-0}{2-0} = 5$



Observe that :

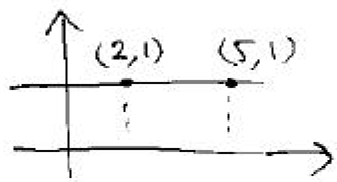


the slope of a line rotated counterclockwise is positive and continues to increase as it gets closer to the y-axis



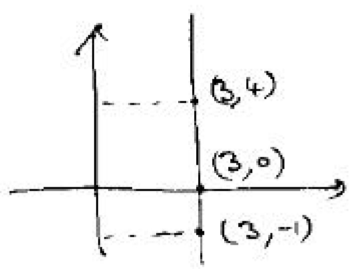
if we rotate clockwise a line starting from the x-axis in the second quadrant we get lines with negative slopes and the values of the slopes get smaller and smaller .

Horizontal lines :



all points on that line have ($y = 1$)

Vertical lines :

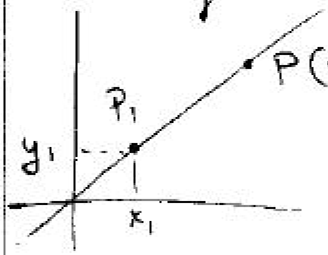


all points on that line satisfy ($x=3$)

* In general , what's the equation of an arbitrary line?

* What are the relations among parallel or perpendicular lines ?

* point-slope form of a line through $P_1(x_1, y_1)$ and with slope m



$m = \text{slope} = \dfrac{y - y_1}{x - x_1}$

so rewrite and get an equation $\boxed{y - y_1 = m(x - x_1)}$

* This might not be so pretty so we can rewrite it as: $y = mx + \underbrace{(y_1 - mx_1)}_{b}$

Such a form $\boxed{y = mx + b}$ is called the slope-intercept form.

* It might happen that m = rational number so we would be tempted to clear denominator

e.g. $y = -\frac{3}{4}x + 5 \iff 4y = -3x + 20$

or $\iff 3x + 4y = 20$

* Thus an equation of a line can be given also in the form

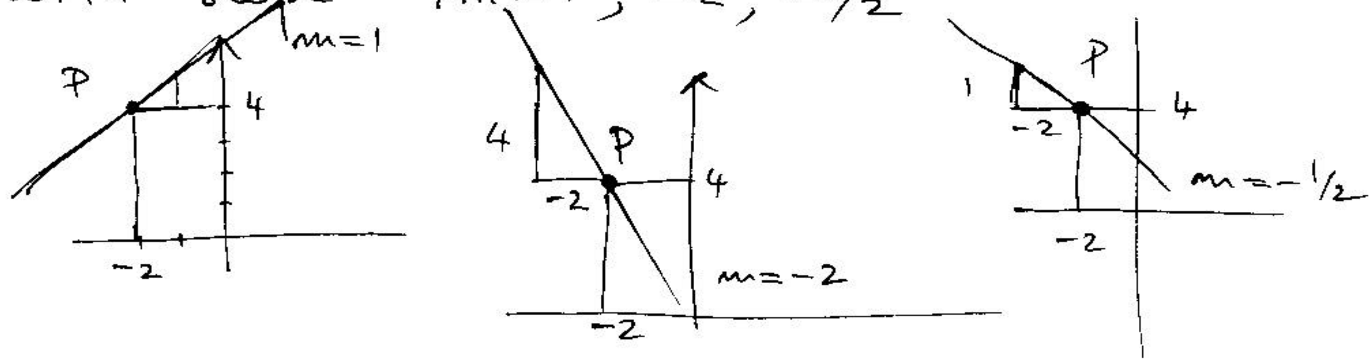
$$\boxed{ax + by = c}$$

$\updownarrow$

Note: if this is the case $y = -\frac{a}{b}x + \frac{c}{b}$

so the slope is $-a/b$ and the y-intercept is c/b.

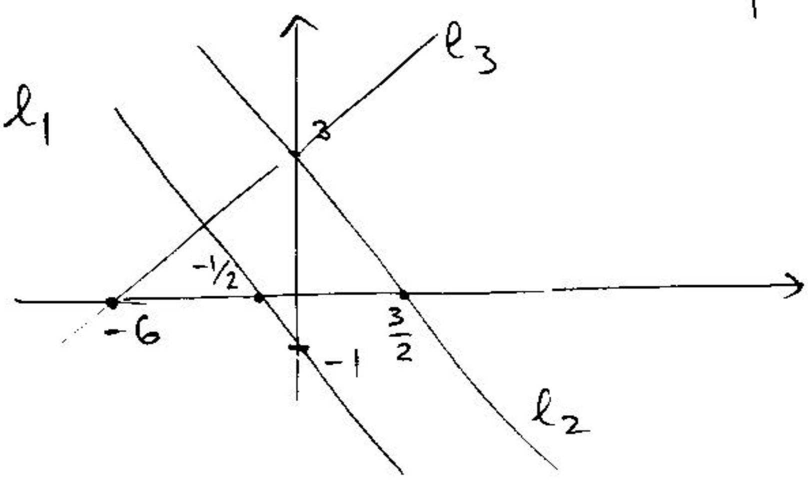
* Sketch the line through $P(-2,4)$ and with slope $m = 1, -2, -1/2$



* Sketch the graph of the lines

l_1: $y = -2x - 1$ l_2: $y = -2x + 3$ l_3: $y = \frac{1}{2}x + 3$

on the same coordinate plane.



l_1 and l_2 are parallel and both perpendicular to l_3

Notice that $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are parallel $\iff m_1 = m_2$

perpendicular $\iff m_1 m_2 = -1$ OR $m_1 = -\dfrac{1}{m_2}$

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* Given $A(5,-3)$ and $m=4$ write an equation for the line

$$(y-(-3)) = 4(x-5) \qquad \text{or}$$

$$y+3 = 4(x-5) \qquad\qquad y = 4x-23$$

* Write an equation for the line through $A(-2,1)$ and $B(3,7)$

$$\text{slope} = m = \frac{7-1}{3-(-2)} = \frac{6}{5}$$

using A: $\quad y-1 = \frac{6}{5}(x+2)$

$$\rightsquigarrow \quad y = \frac{6}{5}x + \frac{12}{5} + 1 \rightsquigarrow y = \frac{6}{5}x + \frac{17}{5}$$

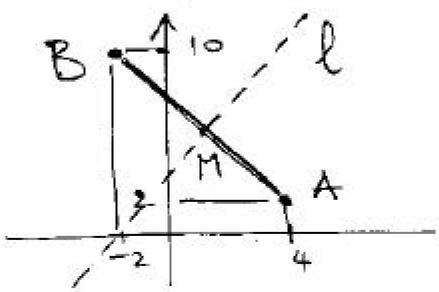
$$\rightsquigarrow \quad 5y - 6x = 17$$

using B: $\quad y-7 = \frac{6}{5}(x-3)$ hmm.... ?

it looks different. But if we expand

$$y = \frac{6}{5}x - \frac{18}{5} + 7 \rightsquigarrow y = \frac{6}{5}x + \frac{17}{5}$$

* Write an equation for the bisector of $A(4,2)$ and $B(-2,10)$



$$M = \text{midpoint} = \left(\frac{4-2}{2}, \frac{2+10}{2}\right) = (1, 6)$$

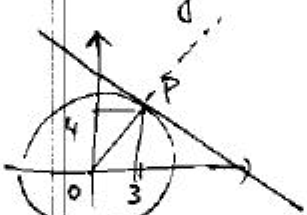
$$\text{slope } AB = \frac{10-2}{-2-4} = \frac{8}{-6} = -\frac{4}{3}$$

$\therefore$ slope of bisector $= -\dfrac{1}{-4/3} = \dfrac{3}{4}$

$\therefore$ equation $y - 6 = \frac{3}{4}(x-1)$

$\rightsquigarrow y = \frac{3}{4}x - \frac{3}{4} + 6 \rightsquigarrow y = \frac{3}{4}x + \frac{21}{4}$.

* Find an equation for the line that is tangent to $x^2+y^2=25$ at $P(3,4)$



slope of $OP = 4/3$

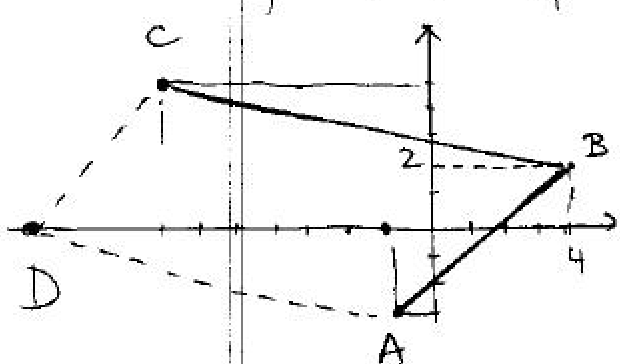
tg line is perpendicular to OP

$\therefore$ slope of tg line $= -3/4$

$\therefore$ eq. of tg line $\quad y - 4 = -\frac{3}{4}(x-3)$

OR $\quad y = -\frac{3}{4}x + \frac{9}{4} + 4 \quad$ OR $\quad y = -\frac{3}{4}x + \frac{25}{4}$

* If 3 consecutive vertices of a parallelogram are $A(-1,-3)$, $B(4,2)$ and $C(-7,5)$ find the fourth vertex.



Let $D(x,y)$ the point we need to find.

slope $DC = \dfrac{y-5}{x+7} =$

$=$ slope $AB = \dfrac{2+3}{\cdot} = 1$

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$\therefore \quad y-5 = x+7 \quad$ or $\quad y = x+12$

slope $AD = \frac{y+3}{x+1} =$ slope $BC = \frac{5-2}{-7-4} = \frac{3}{-11}$

or $\quad y+3 = -\frac{3}{11}(x+1)$

or $\quad y = -\frac{3}{11}x - \frac{3}{11} - 3 \quad$ or $\quad y = -\frac{3}{11}x - \frac{36}{11}$

So we have to satisfy these 2 relations

$$\begin{cases} y = x+12 \\ y = -\frac{3}{11}x - \frac{36}{11} \end{cases} \rightsquigarrow$$

$$x+12 = -\frac{3}{11}x - \frac{36}{11} \quad \text{or}$$

$$11x + 132 = -3x - 36 \longrightarrow 14x = -168$$

$\therefore \quad x = -\frac{168}{14} = -12 \qquad$ and $y = 0$

$\therefore \quad D(-12, 0)$. !!!