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## 2.6 Quadratic functions

We want to discuss a method to graph a quadratic function

$$y = ax^2 + bx + c \qquad a \neq 0 \ ; a, b, c \in \mathbb{R}$$

We'll use shifting and stretching techniques once we manage to express the quadratic expression as

$$y = a(x-h)^2 + k$$

This will be achieved by means of completing the square.

Notice that $V(h,k)$ corresponds to the vertex of the parabola

Let's actually see the exact values for h and k.

$$y = ax^2 + bx + c$$

$$= a\left[x^2 + \frac{b}{a}x\right] + c$$

$$= a\left[x^2 + 2\,\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$

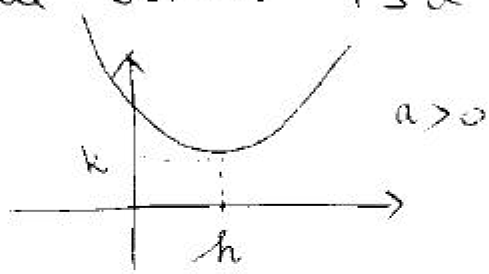
$$= a\left[x^2 + 2\,\frac{b}{2a}x + \frac{b^2}{4a^2}\right] - \frac{b^2}{4a} + c$$

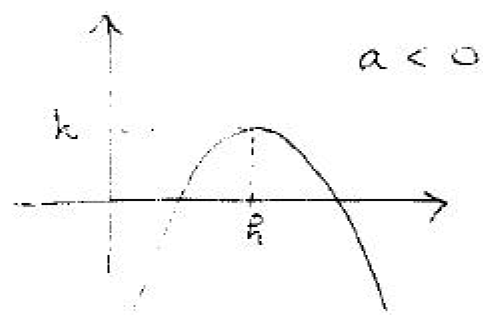
$$= a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$\therefore$ V = vertex of parabola $= \left(-\frac{b}{2a},\ c-\frac{b^2}{4a}\right)$

If $a>0$ the parabola opens up so that the vertex is a minimum.

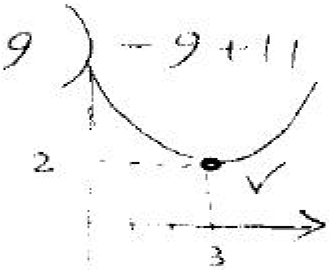
If $a<0$ the parabola opens down so that the vertex is a maximum.




* Express $f(x) = x^2 - 6x + 11$ in the form $a(x-h)^2 + k$.

$$f(x) = x^2 - 6x + 11 = (x^2 - 6x + 9) - 9 + 11$$
$$= (x-3)^2 + 2$$




$V(3,2)$

* Express $f(x) = 5x^2 + 20x + 17$ in the form $a(x+h)^2 + k$.

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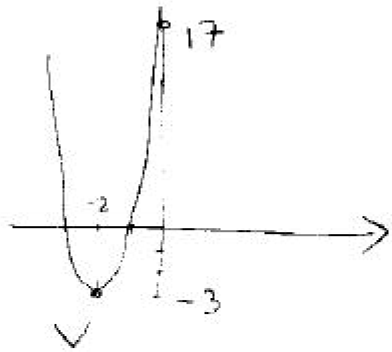
$$f(x) = 5\left(x^2+4x \quad\right)+17$$
$$= 5\left(x^2+4x+4\right)-20+17$$
$$= 5\left(x+2\right)^2-3$$

$\therefore$ $V=(-2,-3)$




* Use the quadratic formula to
(a) find the zeroes of $f(x) = 2x^2-4x-11$
(b) find the max or min. of $f(x)$
(c) sketch the graph of $f(x)$.

$2x^2-4x-11=0 \qquad x_{1,2} = \dfrac{4 \pm \sqrt{16+88}}{4}$

$$= \frac{4 \pm 2\sqrt{26}}{4} = \begin{cases} 1+\frac{\sqrt{26}}{2} \\ 1-\frac{\sqrt{26}}{2} \end{cases}$$

Notice that $f(x) = 2\left(x^2-2x\right)-11$
$$= 2\left(x^2-2x+1\right)-2-11$$
$$= 2\left(x-1\right)^2-13$$

$\therefore$ $V(1,-13)$

On the other hand also notice that

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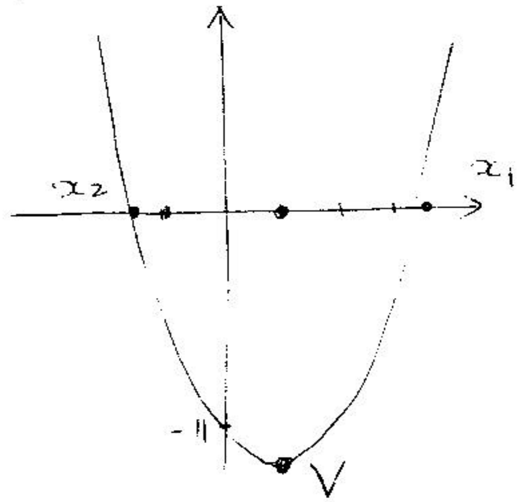
$x_V$ is the midpoint of the zeroes of $f(x)$.

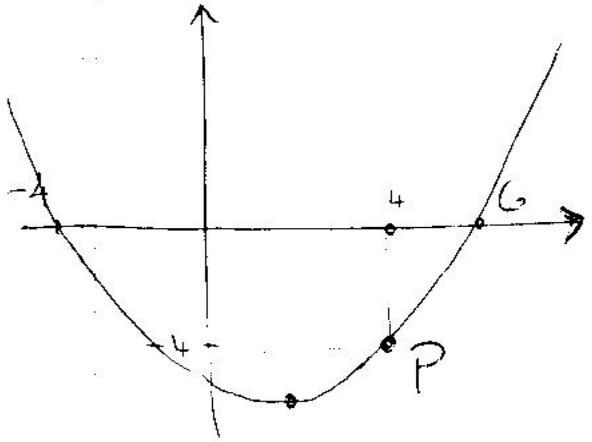
$$x_V = \frac{\left(1 + \frac{\sqrt{26}}{2}\right) + \left(1 - \frac{\sqrt{26}}{2}\right)}{2} = 1 \checkmark$$

$$y_V = f(x_V) = 2(1)^2 - 4\cdot 1 - 11 = -13$$




V is a minimum!

* Find an equation for the parabola of the form $y = a(x - x_1)(x - x_2)$




<u>Well</u>

$$y = a(x - (-4))(x - 6)$$
$$= a(x+4)(x-6)$$

and $P(4, -4)$ satisfies the equation

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$\therefore \quad -4 = a\,(4+4)\,(4-6)$

$-4 = a \cdot 8(-2) \qquad \therefore \ a = 1/4$

$\therefore \quad y = \frac{1}{4}(x+4)(x-6)$

* Find the equation of a parabola that has a vertical axis and satisfies

$V = \text{vertex} = (0,-2)$ and goes through

$P = (3, 25)$

$\therefore \quad y = a(x-0)^2 - 2 = a\,x^2 - 2$

also $\quad 25 = a \cdot (3)^2 - 2 \qquad \therefore \ a = \frac{27}{9} = 3$

$\therefore \quad y = 3x^2 - 2$

* Suppose the parabola has vertex $V(4,-7)$ and has x-intercept at $-4$.

i.e. $P(-4,0)$ is on the parabola

$y = a(x-4)^2 - 7 \longleftarrow$ since $V = (4,-7)$

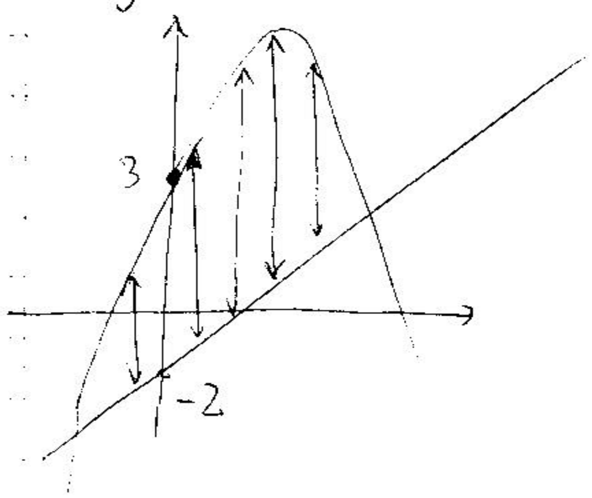
Now $0 = a(-4-4)^2 - 7$

$\therefore \quad a = 7/64$

$\therefore \quad \boxed{y = \frac{7}{64}(x-4)^2 - 7}$

Find the maximum vertical distance $d$ between the parabola

$y = -2x^2 + 4x + 3$ and the line $y = x - 2$




$$d = (-2x^2 + 4x + 3) - (x - 2)$$
$$= -2x^2 + 3x + 5$$

we need to maximize this vertical distance. I.e. we need to locate its maximum:

max dist $= \frac{49}{8} = 6.125$

$$d = -2\left(x^2 - \frac{3}{2}x \quad\right) + 5$$
$$= -2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) + 5$$
$$= -2\left(x - \frac{3}{4}\right)^2 + \frac{9}{8} + 5 = -2\left(x - \frac{3}{4}\right)^2 + \frac{49}{8}$$