MA 113 - Calculus I
FINAL EXAM
Spring 2002
04/29/2002
Name: ____________________  Sec.: __

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Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer
  (unsupported answers may receive NO credit).

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1. Compute the following limits. Each limit is worth 4 points.

(a) \[ \lim_{x \to 2} \frac{x^3 - 1}{x - 1} = \lim_{x \to 2} \frac{(x^2 + x + 1)}{2} = 7 \]

We can write \( x^3 - 1 = (x-1)(x^2 + x + 1) \)

(b) \[ \lim_{x \to 0} \frac{\cos^2 x - 1}{2x^2} = \lim_{x \to 0} \left( -\frac{1}{2} \frac{\sin^2 x}{x^2} \right) = \left( \lim_{x \to 0} \frac{\sin x}{x} \right)^2 = \frac{1}{2} \]

\[ \sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \cos^2 x - 1 = -\sin^2 x \]

(c) \[ \lim_{x \to \infty} \cot \left( \frac{2}{x} + \frac{\pi}{4} \right) = \cot \left( \frac{\pi}{4} \right) = 1 \]

\[ \cot x = \frac{\cos x}{\sin x} \quad \text{as} \quad x \to \infty \quad \frac{2}{x} + \frac{\pi}{4} \to \frac{\pi}{4} \]

\[ \therefore \cot \left( \frac{\pi}{4} \right) = 1 \]

(d) \[ \lim_{x \to \infty} \frac{\sqrt{9x^4 + 8}}{3x^2 + \sqrt{x}} = \lim_{x \to \infty} \frac{\sqrt{9 + \frac{8}{x^4}}}{3 + \frac{1}{\sqrt{x^3}}} = \frac{3}{3} = 1 \]

\[ \text{pts: } /16 \]
2. Find the derivative of the following functions. Each derivative is worth 4 points. Do not simplify your answers.

(a) If \( f(x) = (2x^8 + 7)(3x^2 + 5x) \) then \( f'(x) = \)

\[
f'(x) = 16x^7(3x^2 + 5x) + (2x^8 + 7)(6x + 5)
\]

(b) If \( f(x) = \sin(\sqrt[3]{x^2}) \) then \( f'(x) = \)

\[
f(x) = \sin(x^{2/3}) \quad \text{use chain rule}
\]

\[
f'(x) = \cos(x^{2/3}) \cdot 2/3 \cdot x^{-1/3}
\]

(c) If \( f(x) = \cos^3(x^3) + (5x^2 - 3)^3 \) then \( f'(x) = \)

\[
f'(x) = 3 \cos^2(x^3) \cdot (-\sin(x^3)) \cdot 3x^2 + 3(5x^2 - 3)^2 \cdot 10x
\]

(d) If \( f(x) = \frac{\sin(x^3 - 1)}{x^3 + 1} \) then \( f'(x) = \)

\[
\text{use quotient rule and chain rule}
\]

\[
f'(x) = \frac{[\cos(x^3 - 1) \cdot 3x^2] (x^3 + 1) - \sin(x^3 - 1) \cdot 3x^2}{(x^3 + 1)^2}
\]

\text{pts: 7/16}
3. A particle is moving on a line such that its position after $t$ hours is $s(t) = -t^2 + t + 2$ measured in miles.

(a) Find the velocity of the particle.

$$v(t) = -2t + 1$$

(b) When does the particle change its direction?

It changes direction when $v(t) = 0$,

i.e. at $t = \frac{1}{2}$

(c) What is the largest distance of the particle to its origin within the first 5 hours.

The position at $t = 0$ is $2$,

the position at $t = \frac{1}{2}$ is $\frac{9}{4}$.

The position at $t = 5$ is $-18$.

Thus the largest distance to its origin is $20$. 

pts: 16
4. Consider the function $f(x) = \frac{x^3}{x^2 - 4}$.

(a) (3pts) Determine the intervals where $f(x)$ is increasing or decreasing. Find the values of $f$ at the local minima and maxima of $f$.

\[ f'(x) = \frac{3x^2(x^2 - 4) - x^3(2x)}{(x^2 - 4)^2} = \frac{x^4 - 12x^2}{(x^2 - 4)^2} \]

\[ f'(x) = 0 \iff x^4 - 12x^2 = 0 \iff x^2(x^2 - 12) = 0 \iff x = 0, x = \pm \sqrt{12} \]

(b) (3pts) Determine the intervals where $f'(x)$ is concave up or down. Find the values of $f$ at its inflection points.

\[ f''(x) = \frac{(4x^3 - 24x)(x^2 - 4) - (x^4 - 12x^2) \cdot 2 \cdot x}{(x^2 - 4)^3} = \frac{8x^3 + 96x}{(x^2 - 4)^3} = \frac{8x(x^2 + 12)}{(x^2 - 4)^3} \]

Sign of $f''$: \[\begin{array}{cccccc}
&\bullet&-&+&0&-&+&\bullet&+
\end{array}\]

(c) (2pts) Find the horizontal and vertical asymptotes of the graph of $f$.

Vertical asymptotes: $x = 2, x = -2$.

No horizontal asymptote.

\[ \lim_{x \to \infty} \frac{x^3}{x^2 - 4} = +\infty \]

\[ \lim_{x \to -\infty} \frac{x^3}{x^2 - 4} = -\infty \]

(d) (2pts) Sketch the graph of $f$. **Make sure** to label the local extrema and the inflection points as well as to include the asymptotes of the graph of $f$.  

\[ \text{pts: /10} \]
5. Find the largest area of a rectangle that can be inscribed in a circle of radius 1.

Notice that the main diagonals of any such rectangle are diameters.

Now, if \( x \) and \( y \) are the dimensions of the rectangle we have that \( x^2 + y^2 = 2^2 = 4 \) thus \( y = \sqrt{4-x^2} \). We need to maximize the function

\[
\text{Area} = xy = x \sqrt{4-x^2} \quad 0 \leq x \leq 2
\]

\[
A' = 1 \cdot \sqrt{4-x^2} + x \cdot \frac{1}{\sqrt{4-x^2}} \left( -\frac{kx}{\sqrt{4-x^2}} \right) = \frac{4-x^2 - x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}
\]

\[
A' = 0 \iff 4-2x^2 = 0 \quad x^2 = 2 \quad x = \sqrt{2}
\]

\[
\begin{array}{c|c|c}
    x & A(x) & \text{value at the endpoints} \\
    \hline
    0 & 0 & \{} \text{value at the endpoints} \}\{ \\
    2 & 0 & \}\{ \\
    \sqrt{2} & 2 & \text{max}
\end{array}
\]

Notice \( x=\sqrt{2} = y \) \( \therefore \text{it is a square} \)
6. Find the following indefinite integrals. Each problem is worth 4 points.

(a) \( \int (\sqrt{x} + \sin(5x)) \, dx = \frac{2}{3} \sqrt{x} - \frac{1}{5} \cos(5x) + \text{const} \)

\[
\int \sqrt{x} \, dx + \int \sin(5x) \, dx = \frac{2}{3} x^{3/2} + \frac{1}{5} (-\cos(5x)) + \text{const}
\]

\[
x^{3/2} = x\sqrt{x}
\]

(b) \( \int \sqrt{x} \cdot (x^7 - 1) \, dx = \frac{3}{25} x^{\frac{3}{2}} - \frac{3}{4} x^{\frac{3}{2}} + \text{const} \)

\[
\int (x^{\frac{7}{3}} - x^{\frac{1}{3}}) \, dx = \int (x^{\frac{23}{3}} - x^{\frac{1}{3}}) \, dx = \frac{3}{25} x^{\frac{1}{3}} - \frac{3}{4} x^{\frac{1}{3}} + \text{const}
\]

(c) \( \int \frac{\sin x}{\cos^4 x} \, dx = \frac{1}{4} \cot^4 x + \text{const} \)

\[
\text{use } u = \cot x \quad \Rightarrow \quad \int -\frac{du}{u^5} = -\int u^{-5} \, du = -\frac{1}{-4} u^{-4} + \text{const}
\]

\[
= \frac{1}{4} u^{-4} + \text{const}
\]

(d) \( \int (x^5 + x^3)^7 \cdot (x^5 + x^3)^2 \, dx = \frac{1}{24} (x^6 + x^3)^8 + \text{const} \)

\[
\text{set } u = x^6 + x^3 \quad \Rightarrow \quad \int u^7 \, du = \int \frac{u^7}{3} \, du = \frac{1}{3} \cdot \frac{1}{8} u^8 + \text{const}
\]

pts: /16

\underline{Note:} there was a typo in problem (d)!!
7. Calculate the following definite integrals. Each problem is worth 4 points.

(a) \[ \int_{0}^{9} (x^2 - \sqrt{x}) \, dx = \frac{225}{3} \]

\[ = \frac{1}{3} x^3 - \frac{2}{3} x^{3/2} \bigg|_{0}^{9} = \frac{1}{3} \cdot 9^3 - \frac{2}{3} \cdot 9^{3/2} - 0 = 243 - 18 \]
\[ = 225 \]

(b) \[ \int_{0}^{1} x^3 \sqrt{1 + 2x^4} \, dx = \frac{3\sqrt{3} - 1}{12} \]

\[ u = 1 + 2x^4 \]
\[ du = 8x^3 \, dx \]
\[ \frac{1}{8} \int \sqrt{u} \, du = \frac{1}{8} \int u^{1/2} \, du = \frac{1}{8} \left[ \frac{2}{3} u^{3/2} \right]_1^{3} \]
\[ = \frac{1}{8} \left[ \frac{2}{3} (3\sqrt{3} - 1) \right] = \frac{3\sqrt{3} - 1}{12} \]

(c) \[ \int_{0}^{1} (2x + 1)^2 \, dx = \frac{13}{3} \]

\[ u = 2x + 1 \]
\[ du = 2 \, dx \]
\[ \int \frac{1}{2} u^2 \, du = \frac{1}{6} u^3 \bigg|_{1}^{3} = \]
\[ = \frac{1}{6} \left[ 27 - 1 \right] = \frac{26}{6} = \frac{13}{3} \]

**pts: /12**
8. (a) Sketch the region that is bounded by the graphs of the functions \( f(x) = 2x - x^2 \) and \( g(x) = x^3 \). Determine the points of intersections of the two curves.

\[
\begin{align*}
\left\{ \begin{array}{l}
y = x^3 \\
y = 2x - x^2 
\end{array} \right. \\
2x - x^2 + x^3 = 0 \\
x(x^2 + x - 2) = 0 \\
x(x + 2)(x - 1) = 0 \\
x = 0, 1, -2
\end{align*}
\]

(b) Compute the area of the region.

\[
\text{Area} = \int_{-2}^{0} (x^3 - (2x - x^2)) \, dx + \int_{0}^{1} ((2x - x^2) - x^3) \, dx
\]

\[
= \int_{-2}^{0} (x^3 - 2x + x^2) \, dx + \int_{0}^{1} (-x^3 + x^2 + 2x) \, dx
\]

\[
= \left[ \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 \right]_{-2}^{0} + \left[ \frac{-1}{4}x^4 - \frac{1}{3}x^3 + x^2 \right]_{0}^{1}
\]

\[
= 0 - \left( \frac{1}{4}(4) + \frac{1}{3}(-8) + 4 \right) + \left( \frac{-1}{4} - \frac{1}{3} + 1 \right)
\]

\[
= \frac{8}{3} + \frac{-3 - 4 + 12}{12} = \frac{32 + 5}{12} = \frac{37}{12}
\]

\text{pts: 7/8}
9. Find the volume of the solid that is obtained by rotating about the x-axis the region bounded by the curve \( x - y^3 = 0 \) and the line \( x = 1 \).

\[
\text{Volume} = \int_{0}^{1} \pi \left[ f(x) \right]^2 \, dx
\]

\[
= \int_{0}^{1} \pi \left[ 3 \sqrt[3]{x} \right]^2 \, dx = \int_{0}^{1} \pi x^{2/3} \, dx
\]

\[
= \left. \pi \cdot \frac{3}{5} x^{5/3} \right|_{0}^{1} = \pi \cdot \frac{3}{5} \cdot 1^{5/3} - 0
\]

\[
= \frac{3}{5} \pi
\]

\[\text{pts: } /8\]