Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may receive NO credit).

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>SCORE</th>
<th>TOTAL</th>
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<td>TOTAL</td>
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</table>
1. Find all the critical values and the absolute maximum and absolute minimum values for

\[ f(x) = 3x^4 - 16x^3 + 18x^2 \]

on the closed interval \(-1 \leq x \leq 4\).

\[ f'(x) = 12x^3 - 48x^2 + 36x \]

\[ f'(x) = 0 \iff x^3 - 4x^2 + 3x = 0 \]

\[ x(x^2 - 4x + 3) = 0 \]

\[ x(x - 3)(x - 1) = 0 \]

\( \therefore x = 0, \ x = 1, \ x = 3 \)

<table>
<thead>
<tr>
<th>X</th>
<th>( f(x) )</th>
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<tbody>
<tr>
<td>-1</td>
<td>37</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>-27</td>
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\[ f(x) \text{ has an absolute max of } 37 \text{ at } x = -1 \]

\[ f(x) \text{ has an absolute min of } -27 \text{ at } x = 3 \]

pts: /10

2. (a) Does the Mean Value Theorem apply to the function \( f(x) = \frac{x+1}{x-1} \) on the interval \( 2 \leq x \leq 3 \)?

Why? If so, find all possible values of \( c \) for which the Mean Value Theorem holds on the given interval.

the function \( f(x) \) is continuous and differentiable on \([2, 3]\). The MVT says that there exists \( c \in (2, 3) \) s.t.

\[ f'(c) = \frac{f(3) - f(2)}{3 - 2} = \frac{2}{1} - \frac{3}{1} = -1 \]

Note that \( f'(x) = \frac{1(x-1) - 1(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} \)

\( \therefore \frac{-2}{(c-1)^2} = -1 \)

\( \therefore (c-1)^2 = 2 \)

\( \therefore c = 1 \pm \sqrt{2} \)

But \( 1 - \sqrt{2} \) is outside the interval.

(b) Same as (a), but on the new interval \( 0.5 \leq x \leq 1.5 \).

the mean value theorem does not apply as the function is not continuous at \( x = 1 \in [0.5, 1.5] \)

pts: /10
3. Consider the function: 
\[ f(x) = x^4(x^2 - 3) = x^6 - 3x^4 \]

Each question is worth 5 points.

(a) Determine the intervals where the graph of \( f(x) \) is increasing or decreasing. Find the values of \( f(x) \) at the local maxima and minima of \( f(x) \).

\[ f'(x) = 6x^5 - 12x^3 = 6x^3(x^2 - 2) = 0 \]

\[ \Rightarrow x = 0 \text{ or } x = \pm \sqrt{2} \]

\[ \begin{array}{c|c|c|c|c}
\text{decr} & \text{incr} & \text{decr} & \text{incr} \\
\hline
- & + & - & + \\
\hline
-\sqrt{2} & 0 & \sqrt{2} & 7 \\
\end{array} \]

\[ \text{local min at } x = \pm \sqrt{2} \]

\[ f(\pm \sqrt{2}) = -4 \]

\[ \text{local max at } x = 0 \]

\[ f(0) = 0 \]

(b) Determine the intervals where the graph of \( f(x) \) is concave up or down. Find the values of \( f(x) \) at the inflection points of \( f(x) \).

\[ f''(x) = 30x^4 - 36x^2 = 6x^2(5x^2 - 6) = 0 \]

\[ \Rightarrow x = 0, \quad x = \pm \sqrt{\frac{6}{5}} \]

\[ \begin{array}{c|c|c|c|c}
\text{conc. up} & \text{conc. down} & \text{conc. up} \\
\hline
-\sqrt{\frac{6}{5}} & - & \sqrt{\frac{6}{5}} & + \\
\hline
\end{array} \]

\[ \text{inflection pts} \]

\( x = \pm \sqrt{\frac{6}{5}} \)

\[ f(\pm \sqrt{\frac{6}{5}}) = -2.592 \]

(c) Sketch the graph of \( f(x) \).

Make sure to label the local maxima, the local minima and the inflection points of \( f(x) \).
4. Without using a calculator, show that the equation

\[ x^{101} + x^1 + x - 1 = 0 \]

has exactly one real root.

Observe that if we consider \( f(x) = x^{101} + x + x - 1 \) defined on \([0, 1]\), then by the Intermediate Value Theorem \( f(x) \) has a root in \((0, 1)\). In fact \( f(x) \) is continuous and \( f(0) = -1 \), while \( f(1) = 2 \).

Suppose that \( f(x) \) has 2 roots \( "a, b" \). I.e. \( f(a) = 0 = f(b) \). Since \( f \) is continuous and differentiable by Rolle's Theorem there exist \( c \in (a, b) \) where \( f'(c) = 0 \). BUT \( f'(x) = 101x^{100} + 51x^5 + 1 \) is never zero.

\[ \therefore \text{there is only one real root} \] pts: 78

5. Show that if \( x > 0 \) then \( x + \frac{4}{x^2} \geq 3 \).

Let \( f(x) = x + \frac{4}{x^2} \) defined on the half line \( x > 0 \). Notice that \( f'(x) = 1 - \frac{8}{x^3} \) also, \( f \) has only one critical value \( \text{for } x > 0 \). \( 0 = f'(x) = \frac{x^3 - 8}{x^3} \iff x^3 - 8 = 0 \iff x = 2 \)

\[ \begin{array}{c|cccc}
\text{sign of } f'(x) & - & + & + & +
\end{array} \] \( \therefore f \) has an absolute min at \( x = 2 \)

\( \therefore f(x) = x + \frac{4}{x^2} \geq f(2) = 2 + \frac{4}{2^2} = 3 \)

pts: /10
6. Each question is worth 5 points.

(a) \[ \lim_{{x \to \infty}} \frac{\sqrt{x} + 3}{3 - 2x} = \]

\[ = \lim_{{x \to \infty}} \frac{\frac{1}{\sqrt{x}} + \frac{3}{x}}{3 - 2} = \frac{0}{-2} = 0 \]

(b) \[ \lim_{{x \to \infty}} \frac{2 \sqrt{1 + 9x^2}}{9 - 16x} = \]

\[ = \lim_{{x \to \infty}} \frac{\frac{2 \sqrt{1 + 9x^2}}{x}}{\frac{9 - 16x}{x}} = \lim_{{x \to \infty}} \frac{2 \sqrt{\frac{1}{x^2} + 9}}{9 - 16} = \frac{2 \sqrt{9}}{-16} = \frac{-6}{16} = -\frac{3}{8} \]

(c) Find the vertical and horizontal asymptotes of the curve

\[ f(x) = \frac{3x^2 + 4}{2 - x^2} \]

Compute \( \lim_{{x \to a^+}} f(x) \) and \( \lim_{{x \to a^-}} f(x) \) for all the values of ‘a’ such that the line \( x = a \) is a vertical asymptote of the given function \( f(x) \).

The equation of the horizontal asymptote is \( y = -3 \) as \( \lim_{{x \to \infty}} \frac{3x^2 + 4}{2 - x^2} = -3 \)

The function has 2 vertical asymptotes at \( x = \sqrt{2}, \; x = -\sqrt{2} \).

\[ \lim_{{x \to \sqrt{2}^+}} f(x) = -\infty \]
\[ \lim_{{x \to \sqrt{2}^-}} f(x) = +\infty \]

\[ \lim_{{x \to -\sqrt{2}^+}} f(x) = +\infty \]
\[ \lim_{{x \to -\sqrt{2}^-}} f(x) = -\infty \]

pts: 15
7. Each problem is worth 5 points.

(a) The graph of a function $f(x)$ is shown. Which graph is an antiderivative of $f(x)$ and why?

(b) Find the most general antiderivative of: $f(x) = x^3 + \sqrt{x} - 2 \cos(2x)$. 

\[
\begin{align*}
\frac{df(x)}{dx} &= \frac{1}{4} x^4 + \frac{2}{3} x^{3/2} - \sin(2x) + \text{Const} \\
&= \frac{1}{4} x^4 + \frac{2}{3} x^{3/2} - \sin(2x) + \text{Const} \\
&= \text{don't forget it}
\end{align*}
\]

pts: /10
8. A swimmer $S$ is in the ocean 100 meters from a straight shoreline. A person $P$ in distress is located on the shoreline 300 meters from the point on the shoreline closest to the swimmer.

If the swimmer can swim 3 meters per second and run 5 meters per second, what path should the swimmer follow in order to reach the person in distress as quickly as possible?

Let $x$ be the distance between $H$ and $L$ ($L =$ landing point): $0 \leq x \leq 300$

We need to minimize the time to go from $S$ to $P$ (via $L$):

$$T(x) = \text{time} = \frac{\sqrt{x^2 + 100^2}}{3} + \frac{300 - x}{5}$$

$$T'(x) = \frac{1}{3} \cdot \frac{2x}{2\sqrt{x^2 + 10000}} - \frac{1}{5} = 0$$

$$\Rightarrow \frac{x}{3\sqrt{x^2 + 10000}} = \frac{1}{5}$$

$$\Rightarrow 5x = 3\sqrt{x^2 + 10000}$$

$$\Rightarrow 25x^2 = 9x^2 + 90000 \Rightarrow 16x^2 = 90000 \Rightarrow x = \pm \sqrt{\frac{90000}{16}}$$

$$x = \pm 75$$

But $x = +75$

Critical value $\{75\}$

End points $\{0, 300\}$

$T(x)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$T(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$93.3$ sec</td>
</tr>
<tr>
<td>$300$</td>
<td>$105.4$ sec</td>
</tr>
</tbody>
</table>

$\text{pts: /10}$
9. The graph of the derivative \( f'(x) \) of a function \( f(x) \) is shown:

Each question is worth 3 points.

(a) On what intervals is \( f(x) \) increasing or decreasing?

\[
\begin{array}{ccccccc}
\text{decreasing} & \text{increasing} & \text{decreasing} & \text{increasing} \\
-\infty & \bullet & - & \bullet & 0 & \bullet & 2 & 4 & \infty
\end{array}
\]

(b) At what values of \( x \) does \( f(x) \) attains a local maximum or minimum?

\( f(x) \) has local max at \( x = 0 \)
\( f(x) \) has local min at \( x = -2, 4 \)

(c) On what intervals is \( f(x) \) concave up or down?

\[
\begin{array}{ccccccc}
\text{concave up} & \text{concave down} & \text{concave up} & \text{concave down} \\
-\infty & 1 & 2 & 3 & 5 & \infty
\end{array}
\]

(d) State the \( x \)-coordinates of the inflection points.

\( f(x) \) has inflection points at the points with \( x \)-coordinates:

\( x = -1, 1, 2, \frac{3}{2}, 5 \)

pts: /12