Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,

- clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may receive NO credit)*.

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1. Given the functions: \( f(x) = \sqrt{x^2 - 1} \) and \( g(x) = \sqrt{1 - x} \), find a formula for the composition functions \((g \circ f)(x)\) and \((f \circ g)(x)\).

\[
(g \circ f)(x) = \sqrt{1 - \sqrt{x^2 - 1}} \\
(f \circ g)(x) = \sqrt{-x}
\]

\[\text{domain} \quad \frac{-\sqrt{2}}{2} \leq x \leq -1 \quad \text{and} \quad 1 \leq x \leq \sqrt{2} \]

\[\text{domain} \quad x \leq 0 \]

\[\text{The domain was not required!} \]

\[\text{pts: } /6\]

2. The graph of the function \( f(x) \) is given below

[Graph of \( f(x) \) shown with values from 0 to 6 on the x-axis and -1 to 2 on the y-axis.

Use the above picture to graph the following functions:

\[f(-x) - 1\]

[Graph of \( f(-x) - 1 \) shown with values from -6 to 0 on the x-axis.

\[f(2x)\]

[Graph of \( f(2x) \) shown with values from 0 to 3 on the x-axis and -1 to 2 on the y-axis.

\[\text{pts: } /6\]
3. Compute the following limits. Each limit is worth 5 points.

Note: Remember to simplify your answers!

(a) \( \lim_{x \to -2} \frac{x^2 + 3x + 2}{x + 2} = \lim_{x \to -2} (x+1) = \boxed{-1} \)

\( \frac{x^2 + 3x + 2}{x + 2} = \frac{(x+2)(x+1)}{(x+2)} = (x+1) \)

(b) \( \lim_{s \to a} \frac{1 - f(s)}{2h(s) + g(s) - 2} = \boxed{\frac{2}{7}} \)

given that: \( \lim_{s \to a} f(s) = -3, \lim_{s \to a} g(s) = 0, \lim_{s \to a} h(s) = 8. \)

\[ 1 - \lim_{s \to a} f(s) \]
\[ \frac{2 \lim_{s \to a} h(s) + \lim_{s \to a} g(s) - 2}{2.8 + 0 - 2} = \frac{1 - (-3)}{4} = \frac{1}{14} \]

(c) \( \lim_{x \to 1} \left[ \frac{1}{x-1} - \frac{2}{x^2 - 1} \right] = \lim_{x \to 1} \frac{1}{x+1} = \boxed{\frac{1}{2}} \)

(hint: give common denominators)

\[ \frac{1}{x-1} - \frac{2}{(x-1)(x+1)} = \frac{x+1 - 2}{(x-1)(x+1)} = \frac{x - 1}{(x-1)(x+1)} \]

\[ = \frac{1}{x+1} \]

(d) \( \lim_{x \to 0^+} \frac{x - 1}{x^2(x + 2)} = -\infty \)

\[ \lim_{x \to 0^-} \frac{x - 1}{x^2(x + 2)} = -\infty \]

\[ \frac{-1}{(0^+)^2 \cdot 2} = -\infty \]

\[ \frac{-1}{(0^-)^2 \cdot 2} = -\infty \]

(e) \( \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = 2 \)

\( \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = 2 \)

\( \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2 + h}{h} = 2 \)

\( \therefore \lim_{h \to 0} 2 + h = 2 \)

\( f'(1) \) when \( f(x) = x^2 \)

so that \( f'(x) = 2x \)

\[ \boxed{\text{pts: } / 25} \]
4. (a) (5 pts.) Find the value of $C$ that makes the function $h(x)$ continuous for all real numbers

$$
h(x) = \begin{cases} 
2x - C & x < 2 \\
-Cx^2 + 10 & 2 \leq x 
\end{cases}$$

(b) (2 pts.) Using the value for $C$ found in part (a), graph the function $h(x)$.

(c) (3 pts.) With your choice for $C$ found in part (a), is the function $h(x)$ differentiable at $x = 2$? Explain.

$\lim_{x \to 2^-} 2x - C = \lim_{x \to 2^+} -Cx^2 + 10$

$\therefore 4 - C = -4C + 10$

$4C - C = 10 - 4$

$3C = 6$

$C = 2$

$h(x) = \begin{cases} 
2x - 2 & x < 2 \\
-2x^2 + 10 & 2 \leq x 
\end{cases}$

The function is not differentiable at $x = 2$ since the derivative does not exist. Coming from the left it would be $\frac{2}{15}$, whereas coming from the right it would be $-8$.

pts: /10
5. Does the equation

\[ 2x^3 - x^2 - 3x + 1 = 0 \]

have a root \( c \) between 1 and 2? Explain. (Note: A calculator solution is not an acceptable answer.)

Let \( f(x) = 2x^3 - x^2 - 3x + 1 \) on \( 1 \leq x \leq 2 \).

This function is continuous on that interval and \( f(1) = 2 - 1 - 3 + 1 = -1 \) & \( f(2) = 16 - 4 - 6 + 1 = 7 \).

By the intermediate value theorem there exists \( 1 < c < 2 \) s.t. \( f(c) = 2c^3 - c^2 - 3c + 1 = 0 \).

pts: 7/5

6. Let \( f(x) \) and \( g(x) \) be the functions whose graphs are shown below.

\[ u(x) = f(x) - 2g(x) \]
\[ v(x) = f(x)g(x) \]

Define the functions and find:

\[ u'(5) = \boxed{-\frac{5}{3}} \]
\[ v'(1) = 2 \cdot 1 + 2 \cdot (-1) = \boxed{0} \]

\[ u'(5) = f'(5) - 2g'(5) \]
\[ v'(1) = f'(1)g(1) + f(1)g'(1) \]
\[ f'(5) = -\frac{1}{3} \]
\[ g'(5) = \frac{2}{3} \]
\[ f(1) = 2 \]
\[ g(1) = 1 \]
\[ g'(1) = -1 \]

pts: 7/8
7. Each problem is worth 5 points. Do not simplify your answers in (b) – (d).

(a) Find the equation of the tangent line to the graph of the function \( f(x) = x - x^2 \) at the point on the graph with \( x = 2 \).

\[
\frac{\phi'(x)}{2} = 1 - 2x \quad \frac{\phi'(2)}{2} = -3 \quad \frac{\phi(2)}{2} = 2 - 4 = -2
\]

\[
(y + 2) = -3(x - 2) \quad \text{on} \quad y = -3x + 4
\]

(b) If \( F(u) = \frac{1+u^2}{1-u^2} \) then \( F'(u) = \frac{4u}{(1-u^2)^2} \)

\[
\frac{2u(1-u^2) - (1+u^2)(-2u)}{(1-u^2)^2} \quad = \quad \frac{2u - 2u^3 + 2u + 2u^3}{(1-u^2)^2}
\]

(c) If \( g(x) = (2x^2+1)(2-x^2-\frac{1}{4}x^4) \) then \( g'(x) = \frac{4x(2-x^2-\frac{1}{4}x^4)+(2x^2+1)(-2x-\frac{1}{4}x^3)}{2x^2+1} \)

(d) If \( h(x) = \frac{x^2-\sqrt{x}+1}{\sqrt{x}} \) then \( h'(x) = \frac{\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}}{2x} = \frac{3x^2-1}{2x\sqrt{x}} = \frac{(x-\frac{1}{2\sqrt{x}})\sqrt{x}}{x} \)

\[
h(x) = \frac{x^2}{\sqrt{x}} - 1 + \frac{1}{\sqrt{x}} = x^{3/2} - 1 + x^{-1/2}
\]

\[
h'(x) = \frac{3}{2} x^{1/2} - \frac{1}{2} x^{-3/2} = \frac{3\sqrt{x}}{2} - \frac{1}{2x\sqrt{x}} = \frac{3x^2-1}{2x\sqrt{x}}
\]

\[
h'(x) = \left(2x - \frac{1}{2\sqrt{x}}\right)\sqrt{x} - (x^2-\sqrt{x}+1)^{\frac{1}{2}} = \frac{2x\sqrt{x} - \frac{1}{2} - \frac{x}{2}\sqrt{x} + \frac{1}{2\sqrt{x}}}{2x\sqrt{x}}
\]

pts: /20
8. For each function graphed in figures (a) – (d)

indicate which of the graphs in figures (i) – (viii) is the graph of its derivative.

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<th>Answers:</th>
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<tbody>
<tr>
<td>( f(x) )</td>
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<tr>
<td>( f'(x) )</td>
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pts: /10
9. A spaceship approaching touchdown on the planet Gzxy has height \( y \) (meters) at time \( t \) (seconds) given by

\[
y = 100 - 100t + 25t^2.
\]

\( (a) \) When does the spaceship hit the ground?
\( (b) \) With what velocity does it hit the ground?

\text{pts: } /10

\textbf{Bonus.} Observe that if \( f(x) = x^4 + x^2 \) then \( f'(x) = 4x^3 + 2x \) is an odd function. Similarly, if \( g(x) = \frac{x^2}{x^2+1} \) then \( g'(x) = \frac{2x}{(x^2+1)^2} \) is again an odd function. These and other examples suggest that the derivative of an even function is an odd function.

Using the definition of derivative, prove that if \( f(x) \) is an even function then its derivative \( f'(x) \) is an odd function.

\[
\begin{align*}
f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \\
&= \lim_{-h \to 0} \frac{f(x-h) - f(x)}{-h} = -f'(x)
\end{align*}
\]

\text{pts: } /5