MA 114 - Calculus II
SECOND MIDTERM

Spring 2004
03/09/2004

Name: Answer Key Sec.: ___

Sec. #	Instructor	TAs	Lectures	Recitations
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

Question	Score	Total
1.		54
2.		10
3.		15
4.		15
5.		10
Bonus.		5
Total	(out of 100 pts)	109

1. Evaluate the following integrals. Each problem is worth 7 points.

(*a*) $\int \tan^5 x \sec^3 x\, dx = \boxed{\frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + \text{Const}}$

$\tan^5 x \sec^3 x = \tan^4(x) \sec^2 x \underbrace{\tan x \cdot \sec x}$

also $1 + \tan^2 x = \sec^2 x \rightsquigarrow \tan^2 x = \sec^2 x - 1$

$= (\sec^2 x - 1)^2 \cdot \sec^2 x \tan x \sec x$

set $u = \sec x \quad du = \sec x \tan x\, dx$

$\therefore \int \tan^5 x \sec^3 x\, dx =$

$= \int (u^2-1)^2 u^2\, du$

$= \int (u^4 - 2u^2 + 1) u^2\, du =$

$= \int (u^6 - 2u^4 + u^2)\, du$

$= \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C$

(*b*) $\int \frac{x+2}{x^2+6x+10}\, dx = \boxed{\frac{1}{2} \ln(x^2+6x+10) - \tan^{-1}(x+3) + C}$

$x^2 + 6x + 10 = (x^2+6x+9) + 1 = (x+3)^2 + 1$

$u = x + 3 \quad du = dx \quad x = u - 3$

$\int \frac{x+2}{x^2+6x+10}\, dx = \int \frac{u-1}{u^2+1}\, du$

$= \frac{1}{2} \int \frac{2u}{u^2+1}\, du - \int \frac{1}{u^2+1}\, du =$

$= \frac{1}{2} \ln(u^2+1) - \tan^{-1} u + C$

(*c*) $\int_1^e x^3 \ln x\, dx = \boxed{\frac{3}{16} e^4 + \frac{1}{16}} \approx \boxed{10.300}$

By parts

$\int x^3 \ln x\, dx = \frac{1}{4} x^4 \cdot \ln x - \int \frac{1}{4} x^4 \cdot \frac{1}{x}\, dx = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3\, dx$

$= \frac{1}{4} x^4 \cdot \ln x - \frac{1}{16} x^4 + \text{Const} \qquad \therefore \int_1^e x^3 \ln x\, dx = \frac{1}{4} x^4 \cdot \ln x - \frac{1}{16} x^4 \Big]_1^e$

$= \frac{1}{4} e^4 - \frac{1}{16} e^4 + \frac{1}{16}$

(*d*) $\int \frac{1}{1+\sqrt{x}}\, dx = \boxed{2(1+\sqrt{x}) - 2 \ln|1+\sqrt{x}| + C}$

$\boxed{u = 1 + \sqrt{x}} \qquad du = \frac{1}{2\sqrt{x}}\, dx \rightsquigarrow \boxed{2(u-1)\, du = dx}$

$\sqrt{x} = u - 1$

$\therefore \int \frac{1}{1+\sqrt{x}}\, dx = \int \frac{2u-2}{u}\, du = \int 2\, du - 2 \int \frac{1}{u}\, du$

$= 2u - 2 \ln|u| + C$

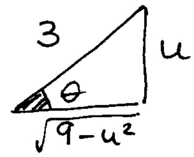
pts: /28

1.(cont.d)

(e) (7 pts) $\int e^x\sqrt{9-e^{2x}}\,dx = \boxed{\frac{9}{2}\cdot \sin^{-1}\left(\frac{e^x}{3}\right) + \frac{\cancel{9}^{\,1}}{2}\,\frac{e^x}{\cancel{3}_{\,1}}\cdot\frac{\sqrt{9-e^{2x}}}{\cancel{3}_{\,1}} + C}$

$u = e^x \quad du = e^x dx$

$= \int \sqrt{9-u^2}\,du$



$\sin\theta = \frac{u}{3} \quad \therefore \quad u = 3\sin\theta$

$\sqrt{9-u^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$= \int 3\cos\theta \; 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = 9\int\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C = \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$

(f) (9 pts) For each of the following functions write out the form of the partial fractions decomposition. DO NOT solve for the coefficients.

$$\frac{x^2+1}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

$$\frac{1}{x^3(x^2+1)} = \frac{1}{x^5+x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}.$$

$$\frac{2x+2}{(x-1)^2} = \frac{2x+2}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

(g) Find the partial fraction decomposition of the function $f(x)$ (5 pts) and then evaluate the corresponding integral (5 pts):

$$f(x) = \frac{1}{(x+1)(x^2+1)} = \boxed{\frac{1/2}{x+1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}},$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)} = \frac{(A+B)x^2+(B+C)x+A+C}{(x+1)(x^2+1)}$$

$\therefore A+B=0 \quad B+C=0 \quad A+C=1 \rightsquigarrow A=-B=C \rightsquigarrow$

$\boxed{A=C=1/2 \quad B=-1/2}$

$$\frac{1}{2}\int\frac{1}{x+1}dx - \frac{1}{2}\int\frac{x-1}{x^2+1}dx = \boxed{\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + C}$$

$$\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x\Big]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1 - 0$$

$$\int_0^1 \frac{1}{(x+1)(x^2+1)}dx = \boxed{\frac{1}{4}\ln 2 + \frac{\pi}{8}} \approx 0.566$$

pts: /26

The **trapezoid rule** T_n and **Simpson's rule** S_n for approximating the integral $\int_a^b f(x)dx$ are:

$$T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots\cdots + 2f(x_{n-1}) + f(x_n)\right),$$

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots\cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)),$$

where $\Delta x = b - a/n$, $x_0 = a$, $x_i = x_0 + i\,\Delta x$ for $i = 1, \ldots, n$, and n is even in Simpson's rule. The **error** in the trapezoid rule, E_T, and in Simpson's rule, E_S, satisfy

$$|E_T| \leq \frac{K_2(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_S| \leq \frac{K_4(b-a)^5}{180n^4}$$

where K_j is a number so that the jth derivative satisfies $|f^{(j)}(x)| \leq K_j$ for all x with $a \leq x \leq b$.

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2. Consider the integral[1]: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx.$

(*a*) Use the trapezoid rule with $n = 4$ to estimate the above integral. Round your answer to 3 decimal places.

$x_0 = 0 \qquad x_1 = \frac{\pi}{8} \qquad x_2 = \frac{\pi}{4} \qquad x_3 = \frac{3\pi}{8} \qquad x_4 = \frac{\pi}{2}$

$$T_4 = \frac{\pi/8}{2} \cdot \left[1 + 2\frac{\sin\left(\frac{\pi}{8}\right)}{\frac{\pi}{8}} + 2\frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} + 2\frac{\sin\left(\frac{3}{8}\pi\right)}{\frac{3}{8}\pi} + \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} \right]$$

$$= \frac{\cancel{\pi}}{16} \cdot \left[\frac{\pi}{\cancel{\pi}} + \frac{16}{\cancel{\pi}}\sin\left(\frac{\pi}{8}\right) + \frac{8}{\cancel{\pi}}\sin(\pi/4) + \frac{16}{3\cancel{\pi}}\sin\left(\frac{3}{8}\pi\right) + \frac{2}{\cancel{\pi}}\sin\left(\frac{\pi}{2}\right) \right]$$

$$\doteq 1.366$$

(*b*) Use Simpson's rule with $n = 4$ to estimate the above integral. Round your answer to 3 decimal places.

$$S_4 = \frac{\pi/8}{3} \cdot \left[1 + 4\frac{\sin\left(\frac{\pi}{8}\right)}{\pi/8} + 2\frac{\sin(\pi/4)}{\pi/4} + 4\frac{\sin\left(\frac{3}{8}\pi\right)}{\frac{3}{8}\pi} + \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} \right]$$

$$= \frac{\cancel{\pi}}{24}\left[\frac{\pi}{\cancel{\pi}} + \frac{32}{\cancel{\pi}}\sin\left(\frac{\pi}{8}\right) + \frac{8}{\cancel{\pi}}\sin(\pi/4) + \frac{32}{3\cancel{\pi}}\sin\left(\frac{3}{8}\pi\right) + \frac{2}{\cancel{\pi}}\sin\left(\frac{\pi}{2}\right) \right]$$

$$\simeq 1.371$$

pts: /10

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[1] Although the notation does not show it explicitly the function integrated is $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

3. (*a*) (5 pts) State the Comparison Theorem for integrals.

look at the notes and/or the book

(*b*) (5 pts) Use the Comparison Theorem to determine whether the following integral converges or diverges

$$\int_1^\infty \frac{\tan^{-1} x}{x^2}\, dx.$$

CONVERGES

Notice $0 \le \dfrac{\tan^{-1} x}{x^2} \le \dfrac{\pi/2}{x^2}$ so that

$$0 \le \int_1^\infty \frac{\tan^{-1} x}{x^2}\, dx \le \frac{\pi}{2}\int_1^\infty \frac{1}{x^2}\, dx$$ converges

(*c*) (5 pts) Use the Comparison Theorem to determine whether the following integral converges or diverges

$$\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\, dx.$$

DIVERGES

$$x = \sqrt{x^2} > \sqrt{x^2 - 0.1}$$

$$\therefore \quad 0 \le \frac{1}{x} < \frac{1}{\sqrt{x^2-0.1}}$$

and $$\underbrace{\int_1^\infty \frac{1}{x}\, dx}_{\text{diverges}} < \int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\, dx$$ as well

pts: /15

4. (*a*) (5 pts) Show that the function $y = e^{-x} + Ce^{-(3/2)x}$ is a solution of the differential equation

$$2y' + 3y = e^{-x}$$

$y' = -e^{-x} - \frac{3}{2}Ce^{-\frac{3}{2}x}$ so that

$2\left(-e^{-x} - \frac{3}{2}Ce^{-\frac{3}{2}x}\right) + 3\cdot\left(e^{-x} + Ce^{-\frac{3}{2}x}\right) =$

$= -2e^{-x} - \cancel{3Ce^{-3/2\,x}} + 3e^{-x} + \cancel{Ce^{-3/2\,x}} = e^{-x}$ ✓

(*b*) (10 pts) Solve the differential equation

$$x\,y\,y' = \ln(x)$$

subject to the initial condition $y(1) = 2$.

$x\,y\,\frac{dy}{dx} = \ln(x) \rightsquigarrow \int y\,dy = \int \frac{1}{x}\ln(x)\,dx$

$u = \ln x$

$du = \frac{1}{x}dx$

$\int y\,dy = \int u\,du \rightsquigarrow$

$\frac{1}{2}y^2 = \frac{1}{2}u^2 + C$ or $y^2 = u^2 + \tilde{C}$

$y^2 = (\ln(x))^2 + \tilde{C}$ or $y = \pm\sqrt{(\ln(x))^2 + \tilde{C}}$

choose the positive solution as $y(1) = 2$.

$2 = \sqrt{(\ln(1))^2 + \tilde{C}}$ $\therefore$ $2 = \sqrt{\tilde{C}}$ $\therefore$ $\tilde{C} = 4$

$\therefore$ $\boxed{y = \sqrt{(\ln(x))^2 + 4}}$

pts: /15

5. Find the length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

from $x = 1$ to $x = 2$.

$y' = \frac{x^2}{2} - \frac{1}{2x^2}$

$1+(y')^2 = 1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2$

$= 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} =$

$= \frac{1}{4x^4}\left(4x^4 + x^8 - 2x^4 + 1\right)$

$= \frac{1}{4x^4}\left(x^8 + 2x^4 + 1\right)$

$= \frac{1}{4x^4}\left(x^4+1\right)^2$

$L = \int_1^2 \sqrt{1+(y')^2}\, dx$

$= \int_1^2 \sqrt{\frac{1}{4x^4}\left(x^4+1\right)^2}\, dx$

$= \int_1^2 \frac{1}{2x^2}\left(x^4+1\right) dx$

$= \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx =$

$= \frac{1}{6}x^3 - \frac{1}{2x}\Big]_1^2 = \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{16-3-2+6}{12} = \boxed{\frac{17}{12}}$

pts: /10

Bonus. Consider the integral $\int_1^2 \frac{1}{x^2}\, dx$.

(*a*) Find n so that the error in approximating the above integral by the trapezoid rule T_n is less than 10^{-4}.

(*b*) Find n so that the error in approximating the above integral by Simpson's rule S_n is less than 10^{-4}.

$f(x) = \frac{1}{x^2} = x^{-2}$ $\quad f'(x) = -2x^{-3}$ $\quad \boxed{f''(x) = 6x^{-4} = \frac{6}{x^4}}$ $\quad f'''(x) = -24x^{-5}$

$\boxed{f^{(IV)}(x) = 120x^{-6} = \frac{120}{x^6}}$

They are both decreasing $\therefore$ the max occurs always at $x=1$.

$|f''(x)| \le \frac{6}{1^4} = 6$ $\quad |f^{(IV)}(x)| \le \frac{120}{1^6} = 120$ on $[1,2]$

$|E_T| \le \frac{6 \cdot 1^3}{12n^2} < 10^{-4}$

$n^2 > \frac{\cancel{6} \cdot 10^4}{\cancel{12}_2}$ $\quad n > \sqrt{5 \cdot 10^3} \cong 70.711$

$\therefore \boxed{n = 71}$

$|E_S| \le \frac{120 \cdot 1^5}{180n^4} < 10^{-4}$

$n > \sqrt[4]{\frac{120 \cdot 10^4}{180}} \cong 9.036$

$\therefore \boxed{n = 10}$ even

pts: /5