MA 114 - Calculus II
THIRD MIDTERM

Spring 2004
04/13/2004

Name: Answer Key Sec.: ___

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		15
2.		5
3.		10
4.		15
5.		15
6.		10
7.		10
8.		15
9.		10
Bonus.		5
TOTAL	out of 100 pts	110

1. (5 pts each) Find the limits of the following sequences

(a) $a_n = \dfrac{\sin(n^2)}{\sqrt{n}}$;

$$-\frac{1}{\sqrt{n}} \le \frac{\sin(n^2)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

$$\searrow \quad 0 \quad \swarrow$$

$\therefore \lim_{n\to\infty} a_n = 0$ by the sandwich theorem

(b) $a_n = \dfrac{4^n}{2^{2n+3}}$;

$$a_n = \frac{4^n}{2^{2n}\cdot 2^3} = \frac{4^n}{4^n \cdot 8} \longrightarrow \frac{1}{8}$$

(c) $a_n = \cos(\pi n)$.

$\lim_{n\to\infty} a_n =$ does not exist

because $\cos(\pi n)$ will keep oscillating between -1 and 1.

pts: /15

2. Use a geometric series to express the repeating decimal $1.\overline{5}$ as a fraction.

$$1.\overline{5} = 1 + \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \cdots$$

$$= 1 + \frac{5}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right)$$

$$= 1 + \frac{5}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1 + \frac{5}{10} \cdot \frac{10}{9} = \boxed{1 + \frac{5}{9} = \frac{14}{9}}$$

pts: /5

3. (5 pts each) Determine if the following series converge. If they do, find their sum:

(b) $\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{4^n}$;

$$\sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n \cdot (-1) = \frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} \cdots$$

$$= \frac{1}{4}\left[1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} \cdots\right] \qquad r = -1/4$$

$$= \frac{1}{4} \cdot \frac{1}{1-(-1/4)} = \frac{1}{4} \cdot \frac{4}{5} = \boxed{1/5}$$

(c) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$.

observe that $a_n = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$

$\therefore$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n =$$

$$= \left[\frac{1}{2} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \left[\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right] + \cdots \left[\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right]$$

$$= \frac{1}{2} - \frac{1}{n+2}$$

$$\therefore \text{sum} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} = \boxed{\frac{1}{2}}$$

pts: /10

4. (5 pts each) Determine whether the following series converge or diverge. Give reasons for your answers.

(a) $\sum_{n=1}^{\infty} \frac{n^2+3}{3n^4+n}$;

it converges by the <u>limit Comparison</u> test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{n^2+3}{3n^4+n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+3n^2}{3n^4+n} = \frac{1}{3}$$

(b) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{(n!)^2}$;

We use the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+3)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n+1)!} = \text{simplify}$$

$$= \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{(n+1)(n+1)} = 4 > 1$$

$\therefore$ the <u>series diverges</u>

(c) $\sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi}{4}\right)\right)^n$.

observe that $\lim_{n\to\infty} \left[\sin\left(\frac{n\pi}{4}\right)\right]^n$ = does not exist

it keeps oscillating between -1 and 1 -

$\therefore$ the series does <u>not</u> Converge -

n	1	2	3	4	5	6	7	8
$\sin\left(\frac{n\pi}{4}\right)$	$+\frac{\sqrt{2}}{2}$	1	$+\frac{\sqrt{2}}{2}$	0	$-\sqrt{2}/2$	-1	$-\frac{\sqrt{2}}{2}$	0

repeat - - - -

pts: /15

5. Determine whether the following series converge absolutely, converge conditionally, or diverge. Give reasons for your answers.

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n^2)}{n^2}$;

We test directly for absolute convergence :

$$0 \le \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin(n^2)|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$$ so it converges absolutely, by direct comparison

(b) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+3}$.

It does not converge absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n+3}$ which does not converge, because of the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.

It converges conditionally, since the conditions of Leibnitz's theorem are met :

- $a_n = \frac{1}{n+3} \ge 0$
- $a_n \longrightarrow 0$ as $n \to \infty$
- $a_{n+1} \le a_n$ for all n

$$\frac{1}{n+4} \overset{?}{\le} \frac{1}{n+3} \iff n+3 \overset{?}{\le} n+4$$

Yes!

pts: /15

6. Determine whether the following series converges or not:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

Will it be of any help if you know the behaviour of the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^3}\ ?$$

Explain....and compute.

$\boxed{\text{Yes,}}$ $f(n) = \frac{1}{n(\ln n)^3}$ when $f(x) = \frac{1}{x(\ln x)^3}$ and

$f(x)$ is positive cont., decreasing $f'(x) = \frac{-(\ln x)^3 - x \cdot 3(\ln x)^2 \cdot \frac{1}{x}}{[x \cdot (\ln x)^3]^2} =$

$= \frac{-(\ln x)^3 - 3(\ln x)^2}{(x(\ln x)^3)^2} < 0$ always.

Hence the series converges $\Leftrightarrow$ the integral does.

$\int_2^{\infty} \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^{+\infty} \frac{du}{u^3} = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u^3} = \lim_{b\to\infty} \left[-\frac{1}{2} \cdot \frac{1}{u^2} \right]_{\ln 2}^{b}$

use $u = \ln x$
$du = \frac{1}{x}\,dx$

$= \lim_{b\to\infty} -\frac{1}{2}\frac{1}{b^2} + \frac{1}{2}\frac{1}{(\ln 2)^2} = \frac{1}{2(\ln 2)^2} < +\infty$

$\therefore$ it is convergent

pts: /10

7. Determine whether the following statements are true (**T**) or false (**F**). Check the appropriate box.

T	**F**	
☑	☐	If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
☐	☑	If $0 \le a_n \le b_n$ for all positive integers n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
☑	☐	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$, then $\lim_{n\to\infty} a_n = 0$.
☑	☐	The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges to $\frac{1}{e} = e^{-1}$.
☐	☑	The Ratio Test can be used to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

pts: /10

8. (*a*) (5 pts) Find the interval of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n!}.$$

we use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right| = \left|\frac{3x}{n+1}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3|x|}{n+1} = 0$$

$\therefore$ it converges for all x $\quad \therefore$ $R = +\infty$

(*b*) (10 pts) Find the series' interval of convergence and, within this interval, the sum $f(x)$ of the series

$$\sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{4^n} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

$$= \sum_{n=1}^{\infty} \left(\frac{(x-2)^2}{4}\right)^n = \frac{(x-2)^2}{4} + \left(\frac{(x-2)^2}{4}\right)^2 + \left(\frac{(x-2)^2}{4}\right)^3 + \dots$$

$$= \frac{(x-2)^2}{4} \cdot \left[1 + \frac{(x-2)^2}{4} + \left(\frac{(x-2)^2}{4}\right)^2 + \dots\right]$$

$$= \frac{(x-2)^2}{4} \cdot \frac{1}{1 - \frac{(x-2)^2}{4}} = \boxed{\frac{(x-2)^2}{4-(x-2)^2}} = \boxed{\frac{x^2-4x+4}{4x-x^2}}$$

it converges when

$$\left|\frac{(x-2)^2}{4}\right| < 1 \iff \left|\frac{x-2}{2}\right| < 1 \iff \boxed{|x-2| < 2}$$

or $\boxed{0 < x < 4}$ it does not converge at the end points

pts: /15

9. Find a power series representation for the function $f(x) = \frac{x^3}{1+x}$ and determine the radius of convergence.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \text{ ----} \quad \text{for} \quad \underline{\underline{|x| < 1}}$$

$$\Rightarrow -1 < x < 1$$

so

$$\frac{x^3}{1+x} = x^3 - x^4 + x^5 - x^6 + x^7 \text{ --- —}$$

$$\boxed{= \sum_{n=3}^{\infty} (-1)^{n-1} x^n}$$

pts: /10

Bonus. Use series to evaluate the limit

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \boxed{\frac{1}{5!} = \frac{1}{120}}$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{x^5}{5!} - \frac{x^7}{7!} \text{ ----}$$

$$\therefore \sin x - x + \frac{1}{6}x^3 = \frac{x^5}{5!} - \frac{x^7}{7!} \text{ ---—}$$

$$\therefore \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x \to 0} \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} \text{ ---} = \frac{1}{5!}$$

Why not check up all your work?

pts: /5