MA 114 - Calculus II
PRACTICE
FINAL EXAM

Spring 2004
05/07/2004

Name: Answer Key Sec.: ____

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		10
2.		30
3.		10
4.		10
5.		15
6.		10
7.		10
8.		10
9.		10
Bonus.		5
TOTAL	out of 100 pts	120

1. Find dy/dx for each of the following functions:

(a) $y = \tan^{-1}(e^{3x})$ $dy/dx =$ $\dfrac{3e^{3x}}{1+e^{6x}}$

$$\frac{1}{1+(e^{3x})^2} \cdot e^{3x} \cdot 3$$

logarithmic differentiation

(b) $y = x^{\sin x}$ $dy/dx =$ $x^{\sin x}\left[\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right]$

$$\ln y = \sin x \cdot \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = y\left[\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right]$$

note that $= \ln(2\ln(x)) = \ln 2 + \ln(\ln(x))$

(c) $y = \ln(\ln(x^2))$ $dy/dx =$ $\dfrac{1}{x \ln x}$

$$\frac{dy}{dx} = \frac{1}{\ln(x^2)} \cdot \frac{1}{x^2} \cdot 2x = \frac{1}{2\ln(x)} \cdot \frac{2x}{x^2}$$

$$= \frac{1}{x \cdot \ln x}$$

pts: /10

2. Evaluate the following integrals. Each problem is worth 5 points.

(a) $\int \frac{e^{2x}}{1+e^{2x}}\,dx =$ $\boxed{\frac{1}{2}\ln\left(1+e^{2x}\right)+C}$

$u = 1+e^{2x} \quad du = e^{2x}\cdot 2\,dx$

$$\therefore \int \frac{e^{2x}}{1+e^{2x}}\,dx = \int \frac{1}{2}\,\frac{du}{u} = \frac{1}{2}\ln|u|+C$$

no need of abs. value

$$= \frac{1}{2}\ln\left|1+e^{2x}\right|+C$$

(b) $\int \frac{x^2+2x+3}{(x^2+1)(x+1)}\,dx =$ $\boxed{\ln|x+1| + 2\tan^{-1}x + C}$

partial fractions

$$\frac{x^2+2x+3}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)(x^2+1)} =$$

$$= \frac{(A+B)x^2+(B+C)x+A+C}{(x^2+1)(x+1)}$$

$\therefore$ $A+B=1 \quad B+C=2 \quad A+C=3$

$B=1-A \quad 1-A+3-A=2 \quad C=3-A$

$-2A = 2-4 \quad \therefore \boxed{A=1},\ \boxed{B=0},\ \boxed{C=2} \quad \therefore \int \frac{1}{x+1}\,dx + \int \frac{2}{1+x^2}\,dx$

(c) $\int x^2 \ln x\,dx =$ $\boxed{\frac{1}{3}x^3\cdot\left[\ln x - \frac{1}{3}\right]+C}$

Integration by parts

$$= \frac{1}{3}x^3\cdot\ln x - \int \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3\cdot\ln x - \int \frac{1}{3}x^2\,dx$$

$$= \frac{1}{3}x^3\cdot\ln x - \frac{1}{9}x^3 + C$$

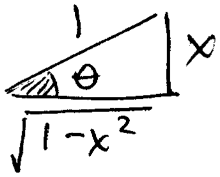
pts: /15

$$= \frac{1}{3}x^3\left[\ln x - \frac{1}{3}\right]+C$$

2.(cont.d)

trig. substitution

(d) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \boxed{\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{(1-x^2)} + C}$



$\begin{cases} dx = \cos\theta\, d\theta \\ x = \sin\theta \end{cases}$ $\quad 1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$

$\boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$

$$\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\, d\theta = \int \sin^2\theta\, d\theta = \int \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2}\theta - \frac{\sin(2\theta)}{4} + C = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$$

(e) $\int x\sqrt{1+x}\,dx = \boxed{2(1+x)\sqrt{1+x}\left(\frac{3x-2}{15}\right) + C}$

$u = 1+x \quad du = dx \quad x = u-1$

$$= \int (u-1)\sqrt{u}\, du = \int (u^{3/2} - u^{1/2})\, du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}u^2\sqrt{u} - \frac{2}{3}u\sqrt{u} + C = 2u\sqrt{u}\left[\frac{u}{5} - \frac{1}{3}\right] + C$$

(f) $\int \frac{3x^4+2x^2+x-1}{1+x^2}\,dx = \boxed{x^3 - x + \frac{1}{2}\ln(1+x^2) + C}$

$$\begin{array}{r|l} & 3x^2 - 1 \\ \hline 1+x^2 & 3x^4 + 2x^2 + x - 1 \\ & 3x^4 + 3x^2 \\ \hline & 0 \quad -x^2 + x - 1 \\ & -x^2 \quad\quad -1 \\ \hline & 0 \quad x \quad 0 \end{array}$$

$$= \int (3x^2 - 1) + \frac{x}{1+x^2}\, dx$$

$$= x^3 - x + \frac{1}{2}\ln(1+x^2) + C$$

pts: /15

$\therefore\ 3x^4 + 2x^2 + x - 1 = (3x^2-1)(x^2+1) + x$

3. An isotope of strontium, Sr^{90}, has a half-life of 25 years.

(a) Find the mass $Q(t)$ of Sr^{90} that remains from a sample of 18 mg after t years.

(b) How long would it take for the mass to decay to 2 mg?

$Q(t) = 18 \cdot \left[\frac{1}{2}\right]^{t/25}$

$t = 25 \frac{\ln(9)}{\ln(2)} = 79.25$ years

$Q(t) = Q_0 e^{rt}$ $\quad Q_0 = 18 \rightsquigarrow Q(t) = 18 e^{rt}$

We know $Q(25) = 9 = \frac{1}{2} 18 \quad \therefore \quad 9 = 18 \cdot e^{25r}$

or $e^{25r} = \frac{1}{2}$ $\quad 25r = \ln(1/2)$

$r = \frac{1}{25} \ln(1/2)$

(a)

$\Rightarrow Q(t) = 18 \underbrace{e^{\ln(\frac{1}{2}) \cdot \frac{t}{25}}}_{\left(e^{\ln(\frac{1}{2})}\right)^{t/25}} = \boxed{18 \left[\frac{1}{2}\right]^{t/25}}$

(b)

$2 = 18 \cdot \left[\frac{1}{2}\right]^{t/25}$

$\ln\left(\frac{1}{9}\right) = t/25 \cdot \ln(1/2)$

$\ldots t = \frac{25 \cdot \ln(9)}{\ln(2)} = 79.25$

pts: /10

4. (5 pts each) Find the limits of the following sequences:

(a) $\quad a_n = \left(1 - \frac{1}{n}\right)^n$;

$a_n = e^{\ln\left(1 - \frac{1}{n}\right)^n} = e^{n \cdot \ln\left(1 - \frac{1}{n}\right)} = e^{\frac{\ln(1 - 1/n)}{1/n}}$

$\therefore \lim_{n\to\infty} a_n = e^{\lim_{n\to\infty} \frac{\ln(1 - 1/n)}{1/n}} = e^{\lim_{x\to 0} \frac{\ln(1-x)}{x}} = \boxed{e^{-1}}$

same as

Use l'Hopital

$\lim_{x\to 0} \frac{\ln(1-x)}{x} = \frac{0}{0} = \lim_{x\to 0} \frac{\frac{-1}{1-x}}{1} = -1$

(b) $\quad a_n = (-1)^n \frac{n+1}{n}$;

limit does not exist

because if n is even $a_n \xrightarrow{n\to\infty} 1$

if n is odd $a_n \xrightarrow{n\to\infty} -1$

pts: /10

Converges conditionally

5. (5 pts each) Determine if the following series are absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\sqrt{2+n}}$;

$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{2+n}}$ does not converge, because of the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ a p-series with $p=\frac{1}{2}\leq 1$

However $\frac{1}{\sqrt{2+n}} \geq 0$, $\frac{1}{\sqrt{2+n}} \to 0$ as $n\to\infty$ an

$\frac{1}{\sqrt{2+(n+1)}} \leq \frac{1}{\sqrt{2+n}}$ for all n $\Longleftrightarrow$ $\sqrt{2+n} \leq \sqrt{3+n}$ ✓

(b) $\sum_{n=1}^{\infty}\frac{n^2}{4^n}$;

let's use the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{4^{n+1}} \cdot \frac{4^n}{n^2} = \frac{1}{4}\underbrace{\frac{(n+1)}{n}}_{\downarrow\, 1} \cdot \underbrace{\frac{(n+1)}{n}}_{\downarrow\, 1} \xrightarrow{n\to\infty} \frac{1}{4}$$

∴ the series converges. It converges absolutely since it has positive terms.

(c) $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^3}$.

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$

but $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges since it is a p-series with $p>1$. Thus our given series converges absolutely because of the direct comparison test.

pts: /15

6. (*a*) Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{3n}{2^n} x^{2n}.$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{3}(n+1)}{2^{n+1}} \cdot x^{2(n+1)} \cdot \frac{2^n}{\cancel{3}n \cdot x^{2n}}\right| = \quad \text{want}$$

$$= \left|\frac{n+1}{n} \cdot \frac{1}{2} \cdot x^2\right| = \frac{1}{2} \frac{n+1}{n} |x|^2 \xrightarrow{n\to\infty} \frac{1}{2}|x|^2 < 1$$

$$\therefore \quad \frac{1}{2}|x|^2 < 1 \implies |x| < \sqrt{2} \qquad \boxed{-\sqrt{2} < x < \sqrt{2}}$$

there is no convergence at the end points!

(*b*) Find the 4th degree Taylor polynomial centered at $a = -1$ for $f(x) = \ln(2+x)$.

$$P_4(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(11)}(a)}{2!}(x-a)^2 + \frac{f^{(111)}(a)}{3!}(x-a)^3 + \frac{f^{(IV)}(a)}{4!}(x-a)^4$$

$$f(x) = \ln(2+x) \qquad f'(x) = \frac{1}{2+x} \qquad f''(x) = -\frac{1}{(2+x)^2} \qquad f'''(x) = \frac{2}{(2+x)^3}$$

$$f^{(IV)}(x) = \frac{-6}{(2+x)^4}$$

$$f(-1) = \ln(2-1) = \ln(1) = 0 \qquad f'(-1) = 1 \qquad f''(-1) = -1$$

$$f'''(-1) = 2 \qquad f^{(IV)}(-1) = -6$$

$$\therefore P_4(x) = 0 + \frac{1}{1}(x+1) + \frac{-1}{2}(x+1)^2 + \frac{2}{3\cdot 2}(x+1)^3 + \frac{-6}{4\cdot 3\cdot 2}(x+1)^4$$

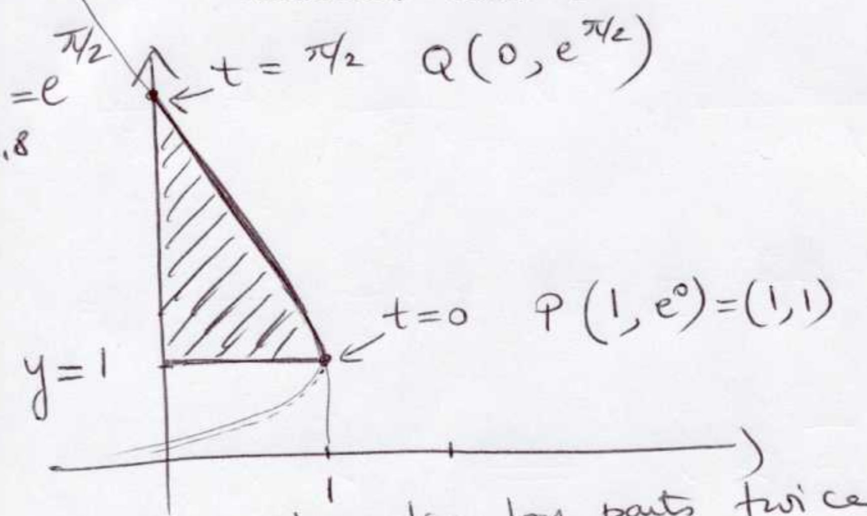
pts: /10

$$= \boxed{(x+1) - \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^4}$$

7. (*a*) Find the area bounded by the curve

$$x(t) = \cos t \qquad y(t) = e^t \qquad 0 \le t \le \pi/2,$$

and the lines $y = 1$ and $x = 0$.



Better find the area as $\int x\, dy =$

$$= \int_0^{\pi/2} \cos(t)\, e^t\, dt =$$

$$= \frac{1}{2}(\cos t + \sin t)e^t \Big]_0^{\pi/2} = \boxed{\frac{e^{\pi/2}}{2} - \frac{1}{2}}$$

integration by parts twice

$$\int \cos t\, e^t\, dt = \cos t\, e^t - \int -\sin t\, e^t\, dt = \cos t\, e^t + \left[\sin t\, e^t - \int e^t \cos t\, dt\right]$$

$$\therefore \quad \int \cos t\, e^t\, dt = \frac{1}{2}(\cos t + \sin t)e^t + C$$

(*b*) Find the length of the curve

$$x(t) = e^t - t \qquad y(t) = 4e^{t/2} \qquad 0 \le t \le 1.$$

$$\frac{dx}{dt} = e^t - 1 \qquad \frac{dy}{dt} = 4e^{t/2} \cdot \tfrac{1}{2} = 2e^{t/2}$$

$$\text{Length} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt = \int_0^1 \sqrt{e^{2t} + 2e^t + 1}\, dt$$

$$= \int_0^1 \sqrt{(e^t + 1)^2}\, dt \int_0^1 (e^t + 1)\, dt = e^t + t \Big]_0^1 =$$

pts: /10

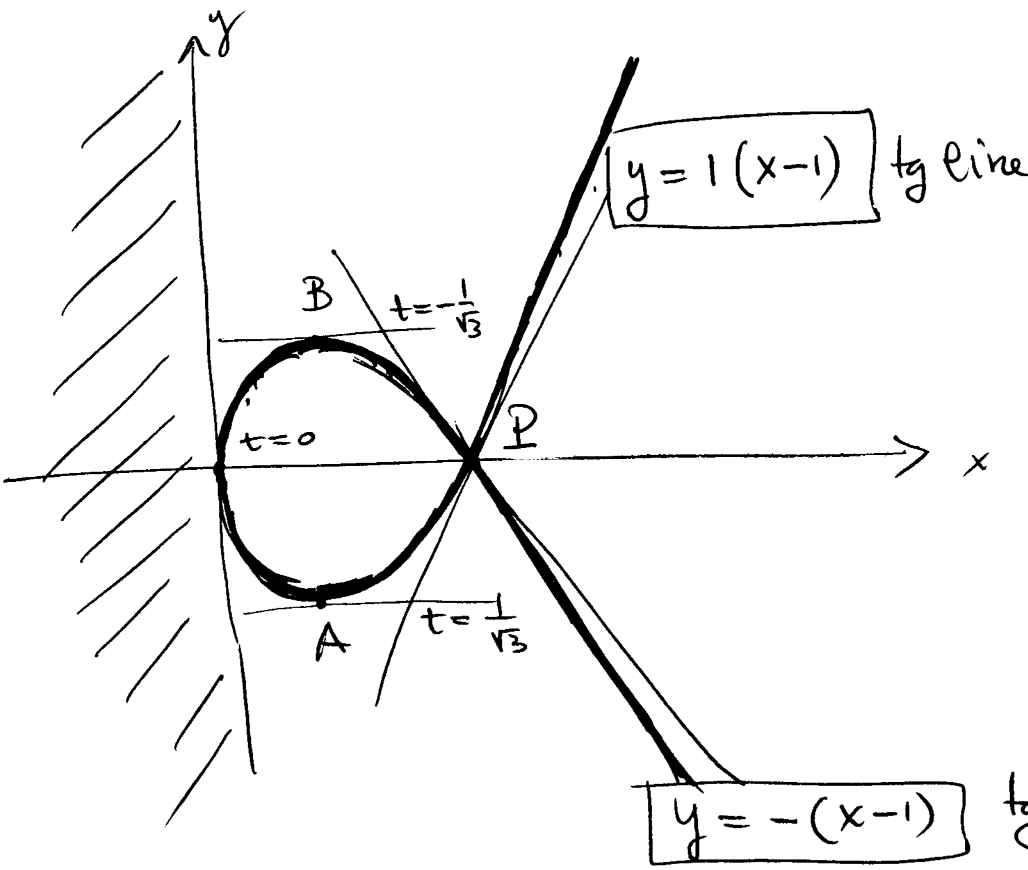
$$= (e + 1) - (e^0 + 0) = \boxed{e}$$

8. (*a*) Sketch the curve

$$x(t) = t^2 \qquad y(t) = t^3 - t$$

(*b*) Find the coordinates of the point where the curve crosses itself.

(*c*) Find the equations of the tangent lines at the point in part (*b*).



Notice that when $t = \pm 1$ y is 0 and $x=1$. So $P(1,0)$ is the point which is described by 2 different choices of the parameter t

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-1}{2t}$$

this is not defined when $t=0$ i.e. at $O(0,0)$ the origin.

$\frac{dy}{dx} = 0$ when $t = \pm\frac{1}{\sqrt{3}}$ $\qquad$ $t = \frac{1}{\sqrt{3}} \rightsquigarrow A = \left(\frac{1}{3}, -\frac{2}{9}\sqrt{3}\right)$

$t = -\frac{1}{\sqrt{3}} \rightsquigarrow B = \left(\frac{1}{3}, \frac{2}{9}\sqrt{3}\right)$

sign of $\frac{dy}{dx}$ = $\quad$ $- - -$ $\;-\frac{1}{\sqrt{3}}\;$ $+ +$ $\;0\;$ $- - -$ $\;\frac{1}{\sqrt{3}}\;$ $+ + +$

pts: /10

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3-1}{2} = \boxed{1} \qquad \left.\frac{dy}{dx}\right|_{t=-1} = \boxed{-1}$$

9. Solve the initial value problem $\dfrac{dy}{dx} = xy + x \qquad y(1) = 0.$

this equation is separable

$$\frac{dy}{dx} = x(y+1) \longrightarrow \frac{dy}{y+1} = x\,dx$$

$$\rightsquigarrow \ln(y+1) = \frac{1}{2}x^2 + C \quad \text{or}$$

$$y+1 = e^{1/2\,x^2 + C} \qquad y = A \cdot e^{1/2\,x^2} - 1 \qquad \text{As } y(1)=0$$

we get $0 = A e^{1/2} - 1 \rightsquigarrow A = e^{-1/2}$

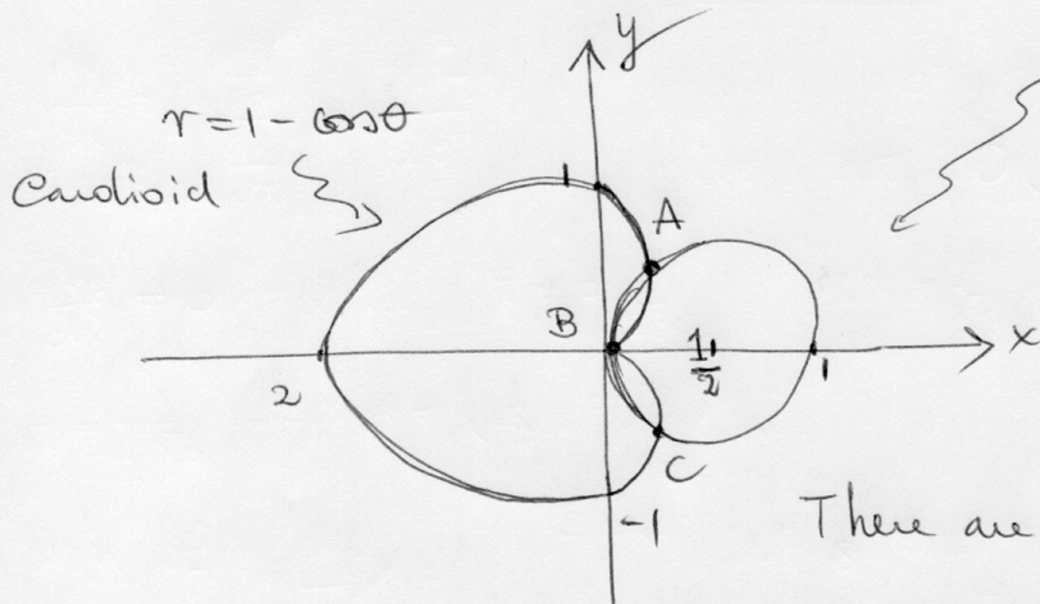
$$\Rightarrow \boxed{y = e^{1/2(x^2-1)} - 1}$$

pts: /10

Bonus. Sketch carefully the graphs of the following equations given in polar coordinates

$$r = 1 - \cos\theta \qquad r = \cos\theta$$

Label and give the coordinates of all the points of intersection.



$$r = \cos\theta$$
$$r^2 = r\cos\theta$$
$$x^2 + y^2 = x$$
$$x^2 - x + y^2 = 0$$
$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \left(\tfrac{1}{2}\right)^2$$

There are 3 points of intersection

Why not check all your work? $r = \boxed{1 - \cos\theta = \cos\theta =} \, r$

pts: /5

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3} \qquad \text{or when } r = 0$$

$A\left(\frac{1}{2}, \frac{1}{2}\right) \qquad C = \left(\frac{1}{2}, -\frac{1}{2}\right) \qquad \hookrightarrow B(0,0)$