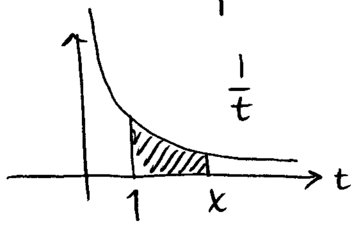
MATH 114 FIRST MIDTERM PRACTICE	Spring 2004 A. Corso	Name: ____________

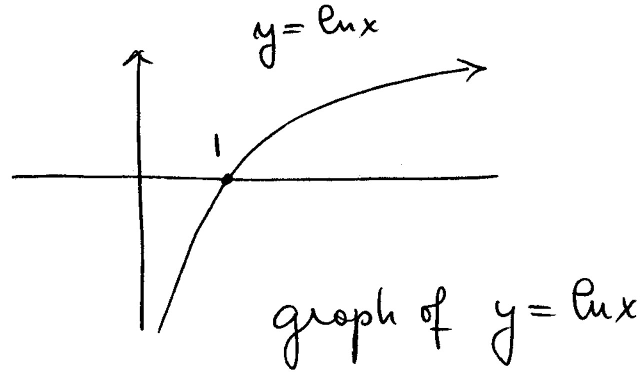
PLEASE, BE NEAT AND SHOW ALL YOUR WORK; CIRCLE YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	15	
2	10	
3	20	
4	25	
5	15	
6	15	
Bonus	5	
TOTAL	100	

1. (*a*) Define the function $\ln x$ for $x > 0$.

$$y = \ln x = \int_1^x \frac{1}{t}\, dt$$





(*b*) If a, b are any positive real numbers and r is a rational number then

* $\ln(ab) = \underline{\ln(a) + \ln(b)}$
* $\ln\left(\frac{a}{b}\right) = \underline{\ln(a) - \ln(b)}$ $\quad \ln\left(\frac{1}{b}\right) = \underline{-\ln(b)}$
* $\ln(a^r) = \underline{r \ln(a)}$

(*c*) Simplify the following expressions

* $\ln \sec\theta + \ln \cos\theta = \underline{0}$

$$= \ln(\sec\theta \cdot \cos\theta) = \ln\left(\frac{1}{\cos\theta} \cdot \cos\theta\right) = \ln(1) = 0$$

* $\left(e^{\ln y - \ln x}\right)^{-1} + \frac{1}{y}\ln(e^x) = \underline{2\frac{x}{y}}$

$$\left[e^{\ln(y/x)}\right]^{-1} + \frac{x}{y} = \left(\frac{y}{x}\right)^{-1} + \frac{x}{y} =$$

$$= \frac{x}{y} + \frac{x}{y} = \frac{2x}{y}$$

pts: /15

2. (*a*) Find $g'(2) = \underline{\ 1/4\ }$ where $g(x)$ is the inverse of the function $f(x) = x^5 - x^3 + 2x$.

We saw in class that $\quad f'(x) = 5x^4 - 3x^2 + 2$

$$g'(a) = \frac{1}{f'(g(a))} =$$

Notice that $g(f(x)) = x$ and if we plug in $x = 1$ $\quad g(f(1)) = 1$ $\quad$ But $f(1) = 1^5 - 1^3 + 2 \cdot 1 = 2$

$\therefore\ g(2) = 1$

$$= \left.\frac{1}{5x^4 - 3x^2 + 2}\right|_{x=1} = \frac{1}{5 - 3 + 2} = \frac{1}{4}$$

(*b*) $\lim_{x \to \infty} \ln(2+x) - \ln(1+x) = \underline{\ 0\ }$.

$$\lim_{x \to \infty} \ln(2+x) - \ln(1+x) =$$

$$= \lim_{x \to \infty} \ln\left(\frac{2+x}{1+x}\right) =$$ as ln is a continuous function

$$= \ln\left(\lim_{x \to \infty} \frac{2+x}{1+x}\right) =$$

$$= \ln\left(\lim_{x \to \infty} \frac{\frac{2}{x} + 1}{\frac{1}{x} + 1}\right) = \ln(1) = 0$$

$\downarrow$

1

pts: /10

3. Find the derivative of the following functions:

(a) $y = \ln(\ln(\ln x))$

$$y' = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

(b) $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$ (use logarithmic differentiation)

$$\ln y = \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left[\frac{1}{x} + \frac{1}{x+1} + \cdots\right]$$

(c) $y = \ln\left(\frac{e^x}{1+e^x}\right)$

smart way $y = \ln e^x - \ln(1+e^x) = x - \ln(1+e^x)$

$$\therefore \frac{dy}{dx} = 1 - \frac{e^x}{1+e^x} = \frac{1+e^x - e^x}{1+e^x} = \frac{1}{1+e^x}$$

otherwise you can use the quotient rule

(d) $y = x\tan^{-1}x + \ln\sqrt{1-x^2}$. $= x\tan^{-1}x + \frac{1}{2}\ln(1-x^2)$

$$y' = \tan^{-1}x + x \cdot \frac{1}{1+x^2} + \frac{1}{\cancel{2}}\frac{1}{1-x^2}\cdot(-\cancel{2}x)$$

$$= \tan^{-1}x + \frac{x}{1+x^2} - \frac{x}{1-x^2}$$

pts: /20

4. Find the following integrals:

(a) $\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos e^{x^2}\,dx$;

set $u = e^{x^2}$ $\quad x=0 \to u=1$

$du = 2x e^{x^2} dx$ $\quad x=\sqrt{\ln\pi} \to u=\pi$

$$= \int_1^{\pi} \cos u\,du = \sin(u)\Big]_1^{\pi} = \sin(\pi) - \sin(1) = \boxed{-\sin(1)}$$

(b) $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$;

set $u = \ln x$ $\quad du = \frac{1}{x}dx$ $\quad x=2 \to u=\ln 2$

$x=16 \to u = \ln 16 = 4\ln 2$

$$= \int_{\ln 2}^{4\ln 2} \frac{du}{2\sqrt{u}} = \sqrt{u}\Big]_{\ln 2}^{4\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$$

(c) $\int \frac{2^{\ln x}}{x}dx$;

$u = \ln x$ $\quad du = \frac{1}{x}du$

$$= \int 2^u du = \frac{2^u}{\ln 2} + C = \boxed{\frac{2^{\ln x}}{\ln 2} + C}$$

(d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos\theta}{1+\sin^2\theta}d\theta$;

$u = \sin\theta$ $\quad du = \cos\theta\,d\theta$

$\theta = -\frac{\pi}{2} \to u=-1$; $\theta = \frac{\pi}{2} \to u=1$

$$\therefore \int_{-1}^{1} \frac{2\,du}{1+u^2} = 2\tan^{-1}u\Big]_{-1}^{1} = 2\left(\tan^{-1}1 - \tan^{-1}(-1)\right) = 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \pi$$

(e) $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$.

$u = \tan y$ $\quad du = \sec^2 y\,dy$

pts: /25

$$\therefore = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \boxed{\sin^{-1}(\tan y) + C}$$

Newton's Law of Cooling states that the rate at which an object changes temperature is proportional to the difference between its temperature and the temperature of the surrounding environment. In other words, if T_S is the (constant) temperature of the environment, the temperature $T(t)$ of the object at time t satisfies

$$\frac{d}{dt}T(t) = k(T(t) - T_S)$$

for some negative constant k. Let T_0 denote the temperature of the object at time $t = 0$. It was shown in class that the function $T(t)$ is given by

$$T(t) = T_S + (T_0 - T_S)e^{kt} \qquad k < 0.$$

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5. A pan of warm water (46°C) was put in a refrigerator. Ten minutes later the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's Law to estimate how cold the refrigerator was.

$T(t) = T_S + (T_0 - T_S)e^{kt}$ $\qquad$ $T_0 = 46°C$

It is known that $\boxed{T(10) = 39\,; \quad T(20) = 33}$

$\therefore$

$39 = T_S + (46 - T_S)e^{10t} \Longleftrightarrow \dfrac{39 - T_S}{46 - T_S} = e^{10t}$

$33 = T_S + (46 - T_S)e^{20t} \Longleftrightarrow \dfrac{33 - T_S}{46 - T_S} = e^{20t}$

Notice that: $e^{20t} = \left(e^{10t}\right)^2$

$\therefore \dfrac{33 - T_S}{46 - T_S} = \left(\dfrac{39 - T_S}{46 - T_S}\right)^2 \Longleftrightarrow (33 - T_S)(46 - T_S) = (39 - T_S)^2$

$1518 - 79T_S + \cancel{T_S^2} = 1521 - 78T_S + \cancel{T_S^2}$

$1518 - 1521 = 79T_S - 78T_S$

$\therefore \boxed{T_S = -3\,°C}$

pts: /15

Note k was not requested!!!

6. Use l'Hôpital's rule to find the following limits:

(a) $\lim\limits_{x \to 0} \frac{e^x - x - 1}{x^2} = \underline{\quad 1/2 \quad}$;

$$\frac{0}{0} \quad = \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x}{2} =$$

$$= \frac{1}{2}$$

(b) $\lim\limits_{x \to 1} \frac{\ln x}{x - 1} = \underline{\quad 1 \quad}$;

$$\frac{0}{0} \quad = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

(c) $\lim\limits_{x \to \infty} \frac{\ln(\ln x)}{\sqrt{x}} = \underline{\quad 0 \quad}$.

$$\frac{\infty}{\infty} \quad = \lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{\ln x \cdot x}$$

$$= \lim_{x \to \infty} \frac{2}{\sqrt{x} \cdot \ln x} = \frac{2}{\infty} = 0$$

pts: /15

Bonus. Choose one of the following questions.

(*a*) Simplify the following expression: $\text{sech}(\ln x) = \underline{\dfrac{2x}{x^2+1}}$;

$$\text{sech}(\theta) = \frac{1}{\cosh\theta} = \frac{2}{e^{\theta}+e^{-\theta}}$$

$$\therefore \text{sech}(\ln x) = \cdots = \frac{2}{e^{\ln x}+e^{-\ln x}} = \frac{2}{x+e^{\ln x^{-1}}} = \frac{2}{x+\frac{1}{x}} = \frac{2x}{x^2+1}$$

(*b*) Find the derivative of the following function: $f(x) = e^{\tanh x}\ln(\sinh x)$;

$$y' = e^{\tanh x}\cdot \text{sech}^2 x \cdot \ln(\sinh x) + e^{\tanh x}\cdot \frac{1}{\sinh x}\cdot \cosh x$$

$$= \boxed{e^{\tanh x}\left[\text{sech}^2 x \ln(\sinh x) + \coth x\right]}.$$

(*c*) Evaluate the following integral: $\int_0^{\ln 2} \tanh(2x)\,dx = \underline{\qquad\qquad}$.

$u = 2x$

$du = 2\,dx$

$$= \int_0^{2\ln 2} \frac{1}{2}\tanh u \, du = \frac{1}{2}\int_0^{\ln 4} \frac{\sinh u}{\cosh u}\, du$$

$w = \cosh u \qquad dw = \sinh u \, du$

$$= \frac{1}{2}\int_1^{17/8} \frac{dw}{w} = \frac{1}{2}\ln w \Big]_1^{17/8} =$$

$$= \frac{1}{2}\ln\left(\frac{17}{8}\right) - \frac{1}{2}\cancel{\ln(1)}_{\searrow 0} = \boxed{\ln\sqrt{\frac{17}{8}}}$$

$$\cosh 0 = \frac{e^0+e^{-0}}{2} = 1$$

$$\cosh(\ln 4) = \frac{e^{\ln 4}+e^{-\ln 4}}{2} = \frac{4+\frac{1}{4}}{2} = \frac{17}{8}$$

pts: /5