MA 114 - Calculus II
PRACTICE
SECOND MIDTERM

Spring 2004
03/09/2004

Name: ______________________ Sec.: ____

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		54
2.		10
3.		15
4.		15
5.		10
Bonus.		5
TOTAL	out of 100 pts	109

1. Evaluate the following integrals. Each problem is worth 7 points.

(a) $\int \sin^3 x \cos^3 x\, dx = \boxed{\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + \text{Const}}$

$= \int \sin^3 x \underbrace{\cos^2 x}_{1-\sin^2 x} \cos x\, dx = \int \sin^3 x (1-\sin^2 x) \cos x\, dx$

$= \int (\sin^3 x - \sin^5 x) \cos x\, dx = \int (u^3 - u^5) du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + \text{Const}$

$u = \sin x \quad du = \cos x\, dx$

$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + \text{Const}$

(b) $\int \frac{x}{x^2+4x+5}\, dx = \boxed{\frac{1}{2}\ln(x^2+4x+5) - 2\tan^{-1}(x+2) + \text{const}}$

$= \int \frac{x}{(x^2+4x+4)+1}\, dx = \int \frac{x}{(x+2)^2+1}\, dx$ set $u = x+2,\ du = dx,\ x = u-2$

Complete squares

$= \int \frac{u-2}{u^2+1}\, du = \frac{1}{2}\int \frac{2u}{u^2+1}\, du - 2\int \frac{1}{u^2+1}\, du = \frac{1}{2}\ln(u^2+1) - 2\tan^{-1} u + \text{Co}$

(c) $\int \sqrt{x}\ln(5x)\, dx = \boxed{\frac{2}{3}x\sqrt{x}\left(\ln(5x) - 2/3\right) + \text{const}}$

By parts

$= \frac{2}{3}x^{3/2}\cdot \ln(5x) - \frac{2}{3}\int x^{3/2}\cdot \frac{1}{5x}\cdot 5\, dx = \frac{2}{3}x^{3/2}\cdot \ln(5x) - \frac{2}{3}\int x^{1/2}\, dx$

$= 2/3\, x^{3/2}\ln(5x) - 4/9\, x^{3/2} + \text{Const} = \frac{2}{3}x\sqrt{x}\left[\ln(5x) - 2/3\right] + \text{const}$

(d) $\int \frac{1+\sin x}{\cos^2 x}\, dx = \boxed{\frac{\sin x + 1}{\cos x} + \text{Const}}$

$= \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx = \int \sec^2 x\, dx - \int \frac{-\sin x}{\cos^2 x}\, dx$

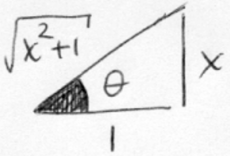
$= \tan x - \left(-\frac{1}{\cos x}\right) + \text{Const}$

$= \frac{\sin x + 1}{\cos x} + \text{Const}$

pts: /28

1.(cont.d)

(*e*) (7 pts) $\int \frac{1}{(x^2+1)^{\frac{3}{2}}}\,dx = \boxed{\frac{x}{\sqrt{x^2+1}} + \text{Const}}$.



$\tan\theta = \frac{x}{1}$ $\therefore$ $x = \tan\theta$ $\quad dx = \sec^2\theta\, d\theta$

$x^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$$= \frac{x}{\sqrt{x^2+1}} + C$$

$$\therefore \int \frac{1}{(x^2+1)^{3/2}}\,dx = \int \frac{1}{(\sec^2\theta)^{3/2}} \cdot \sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\, d\theta \stackrel{!}{=} \sin\theta + C$$

(*f*) (9 pts) For each of the following functions write out the form of the partial fractions decomposition. DO NOT solve for the coefficients.

$$\frac{x}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}.$$

$$\frac{x^2+1}{x^4+x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+2}.$$

irreducible

$x^2(x^2+x+2)$

$\Delta = 1 - 4\cdot 2\cdot 1 = -7 < 0$

$$\frac{x}{x^4+2x^2+1} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

$(x^4+2x^2+1) = (x^2+1)^2$

(*g*) Find the partial fraction decomposition of the function $f(x)$ (5 pts) and then evaluate the corresponding integral (5 pts):

$$f(x) = \frac{1}{x^4+x^2} = \boxed{\frac{1}{x^2} + \frac{-1}{x^2+1}},$$

$x^4+x^2 = x^2(x^2+1)$

$$\frac{1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$$

$\therefore 1 = (A+C)x^3 + (B+D)x^2 + Ax + B$

$A=0,\ B=1$

$\therefore A+C=0 \quad B+D=0 \quad A=0 \quad B=1 \rightsquigarrow \quad C=0,\ D=-1$

$$\int \frac{1}{x^4+x^2}\,dx = \boxed{-\frac{1}{x} - \tan^{-1}x + \text{Const}}.$$

pts: /26

The **trapezoid rule** T_n and **Simpson's rule** S_n for approximating the integral $\int_a^b f(x)dx$ are:

$$T_n = \frac{\Delta x}{2}(f(x_0)+2f(x_1)+\cdots\cdots+2f(x_{n-1})+f(x_n)),$$

$$S_n = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)),$$

where $\Delta x = b-a/n$, $x_0 = a$, $x_i = x_0 + i\Delta x$ for $i = 1,\ldots,n$, and n is even in Simpson's rule.
The **error** in the trapezoid rule, E_T, and in Simpson's rule, E_S, satisfy

$$|E_T| \le \frac{K_2(b-a)^3}{12n^2} \quad \text{and} \quad |E_S| \le \frac{K_4(b-a)^5}{180n^4}$$

where K_j is a number so that the jth derivative satisfies $|f^{(j)}(x)| \le K_j$ for all x with $a \le x \le b$.

2. Consider the integral $\int_0^2 e^{-x^2}\,dx$.

(a) Use the trapezoid rule with $n = 5$ to estimate the above integral. Round your answer to 3 decimal places.

$\Delta x = \frac{2-0}{5} = 0.4 \qquad x_0 = 0,\ x_1 = 0.4,\ x_2 = 0.8,\ x_3 = 1.2,\ x_4 = 1.6,\ x_5 = 2$

$$T_n = \frac{0.4}{2}\left(e^{-0^2} + 2e^{-(0.4)^2} + 2e^{-(0.8)^2} + 2e^{-(1.2)^2} + 2e^{-(1.6)^2} + e^{-2^2}\right)$$

$$\cong \boxed{0.881}$$

(b) Use Simpson's rule with $n = 4$ to estimate the above integral. Round your answer to 3 decimal places.

$\Delta x = \frac{2-0}{4} = 0.5 \qquad x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$

$$S_n = \frac{0.5}{3}\left(e^{-0^2} + 4e^{-(0.5)^2} + 2e^{-1^2} + 4e^{-(1.5)^2} + e^{-2^2}\right)$$

$$\cong \boxed{0.879}$$

pts: /10

3. (*a*) (5 pts) State the Comparison Theorem for integrals.

look at the class notes!

(*b*) (5 pts) Use the Comparison Theorem to determine whether the following integral converge or diverge

$$\int_0^\infty \frac{\sin^2(x)}{1+x^2}\,dx. \qquad \underline{\underline{\text{CONVERGES}}}$$

observe that $\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} \tan^{-1}x\Big]_0^b$

$= \lim_{b\to+\infty} \tan^{-1} b - 0 = \frac{\pi}{2}$ $\therefore$ it converges.

<u>Now</u> $0 \leq \frac{\sin^2(x)}{1+x^2} \leq \frac{1}{1+x^2}$ and since $\int_0^{+\infty} \frac{1}{1+x^2}\,dx$ conv. so does $\int_0^\infty \frac{\sin^2(x)}{1+x^2}\,dx$

(*c*) (5 pts) Use the Comparison Theorem to determine whether the following integral converge or diverge

$$\int_1^\infty \frac{2+e^{-x}}{1+x}\,dx. \qquad \underline{\underline{\text{DIVERGES}}}$$

Observe that $2 \leq 2+e^{-x}$ for $x \geq 0$,

$$\therefore\ \underbrace{2\int_1^\infty \frac{1}{1+x}\,dx}_{\text{diverges}} \leq \int_1^{+\infty} \frac{2+e^{-x}}{1+x}\,dx$$

and diverges. Why? For $1 < x$ we have

$1+x < 2x \rightsquigarrow \frac{1}{2x} < \frac{1}{1+x}$ so that

$$\underbrace{2\int_1^\infty \frac{1}{x}\,dx}_{\text{diverges (seen in class)}} < \int_1^\infty \frac{1}{1+x}\,dx$$

$p=1$

pts: /15

4. A model for a growth function for a limited population is given by the *Gompertz function* which is a solution of the differential equation

$$\frac{dy}{dt} = c\ln\left(\frac{M}{y}\right)y$$

where c is a constant and M is the maximum size of the population.

(*a*) (12 pts) Solve the differential equation.

$$\frac{dy}{\ln\left(\frac{M}{y}\right)\cdot y} = c\,dt \rightsquigarrow \frac{dy}{\left[\ln(M) - \ln(y)\right]y} = c\,dt$$

$$\rightsquigarrow \int \frac{dy}{(\ln y - \ln M)y} = \int -c\,dt \qquad \text{use } u = \ln y \quad du = \frac{1}{y}\,dy$$

$$\rightsquigarrow \int \frac{du}{u - \ln M} = -\int c\,dt \quad \therefore \ln(u - \ln M) = -ct + A$$

$$\therefore \ln y - \ln M = e^{-ct}\cdot e^{A} \quad \text{or} \quad \ln\left(\frac{y}{M}\right) = Be^{-ct}$$

$y(t) =$ ____________ $\rightsquigarrow \boxed{y = M e^{Be^{-ct}}}$

(*b*) (3 pts) Compute $\lim\limits_{t\to\infty} y(t) =$ M

$$\lim_{t\to\infty} y(t) = \lim_{t\to+\infty} M e^{B\overbrace{e^{-ct}}^{\to 0}} = \lim_{t\to+\infty} M e^{B\cdot 0}$$

$$= M\cdot 1 = M$$

pts: /15

5. Find the length of the curve

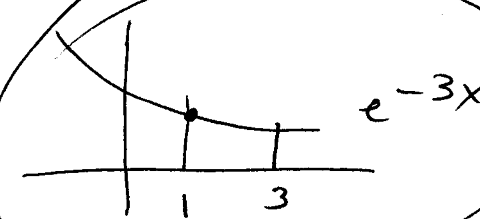
$$y = \int_0^x \sqrt{3t^4 - 1}\, dt$$

from $x = -2$ to $x = -1$.

$$y' = \sqrt{3x^4 - 1} \qquad L = \int_{-2}^{-1} \sqrt{1 + (y')^2}\, dx =$$

$$= \int_{-2}^{-1} \sqrt{\cancel{1} + 3x^4 - \cancel{1}}\, dx - \sqrt{3} \int_{-2}^{-1} x^2\, dx$$

$$= \frac{\sqrt{3}}{3} x^3 \Big]_{-2}^{-1} = \frac{7}{3}\sqrt{3}$$



pts: /10

Bonus. Consider the integral $\int_1^3 e^{-3x}\, dx$.

(*a*) Find n so that the error in approximating the above integral by the trapezoid rule T_n is less than 10^{-4}.

(*b*) Find n so that the error in approximating the above integral by Simpson's rule S_n is less than 10^{-4}.

(a) $f^{(2)}(x) = (-3)^2 e^{-3x}$ $\quad |f^{(2)}(x)| = |(-3)^2 e^{-3x}| \leq 9e^{-3} = K_2$

for all $1 \leq x \leq 3$

$$\therefore |E_T| \leq \frac{9e^{-3} \cdot (3-1)^3}{12 n^2} < 10^{-4} \quad \therefore n^2 > \frac{9 \cdot e^{-3} \cdot 10^4}{12}$$

$$\therefore n > \sqrt{\frac{9 \cdot 10^4}{12 e^3}} \cong 54{,}65 \longrightarrow n = 55$$

(b) $f^{(4)}(x) = (-3)^4 e^{-3x}$ $\quad \therefore |f^{(4)}(x)| \leq 3^4 \cdot e^{-3} = K_4$ $\quad$ $n=10$

pts: /5

$$|E_S| \leq \frac{81 \cdot e^{-3} \cdot (3-1)^5}{180\, n^4} < 10^{-4} \leadsto n > \sqrt[4]{\frac{81 \cdot 10^4 \cdot 2^5}{180 e^{-3}}} \cong 9{,}201$$