MA 114 - Calculus II
PRACTICE
THIRD MIDTERM

Spring 2004
04/13/2004

Name: Answer Key Sec.: ___

SEC.	INSTRUCTORS	T.A.'S	LECTURES	RECITATIONS
001	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 8:00-9:15, CB 347
002	A. Corso	D. Watson	MWF 8:00-8:50, CP 222	TR 12:30-1:45, CP 155
003	A. Corso	S. Petrovic	MWF 8:00-8:50, CP 222	TR 3:30-4:45, CB 347

Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (*unsupported answers may receive NO credit*).

QUESTION	SCORE	TOTAL
1.		15
2.		15
3.		15
4.		15
5.		10
6.		10
7.		15
8.		10
Bonus.		5
TOTAL	out of 100 pts	110

1. (5 pts each) Find the limits of the following sequences

(a) $a_n = (-1)^n \frac{\sin n}{n}$;

observe that $\boxed{-\frac{1}{n} \le (-1)^n \frac{\sin(n)}{n} \le \frac{1}{n}}$

$\to 0 \leftarrow$ as $n \to \infty$

so by the "sandwich" theorem $\lim_{n\to\infty} a_n = 0$

(b) $a_n = \ln(2n) - \ln(3n+1)$;

$= \ln\left(\frac{2n}{3n+1}\right)$

$\therefore \lim_{n\to\infty} a_n = \ln\left[\lim_{n\to\infty} \frac{2n}{3n+1}\right] = \boxed{\ln(2/3)}$

(c) $a_n = \frac{1}{n}\int_1^n \frac{1}{x}dx.$ observe that

$a_n = \frac{\ln n}{n}$ $\lim_{n\to\infty} a_n = \frac{+\infty}{+\infty}$

use l'Hôpital theorem

$\lim_{x\to\infty} \frac{\ln x}{x} = \frac{+\infty}{+\infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

pts: /15

$\therefore \boxed{\lim_{n\to\infty} \frac{\ln n}{n} = 0}$

2. (5 pts each) Determine if the following series converge. If they do, find their sum:

(a) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$;

observe that

$$\frac{1}{n} \le \frac{\ln n}{n} \quad \text{for all } n \geq 2$$

so $\sum_{n=2}^{\infty} \frac{1}{n} \le \sum_{n=2}^{\infty} \frac{\ln n}{n}$ ∴ diverges because of the direct comparison test

↳ diverges

(b) $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{5^n}$; $= \frac{1}{5^2} - \frac{1}{5^3} + \frac{1}{5^4} - \frac{1}{5^5}$ etc ...

$$= \sum_{n=2}^{\infty} \left(-\tfrac{1}{5}\right)^n = \frac{1}{5^2}\left(1 + (-\tfrac{1}{5}) + \left(-\tfrac{1}{5}\right)^2 + \left(-\tfrac{1}{5}\right)^3 + \cdots \right)$$

it is a geometric series with $r = -\frac{1}{5}$ and $|r| = \frac{1}{5} < 1$

∴ it converges to $= \frac{1}{25} \cdot \frac{1}{1-(-\frac{1}{5})} = \frac{1}{25} \cdot \frac{5}{6} = \boxed{\frac{1}{30}}$

(c) $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

It converges for sure use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

What about its sum? It is a telescoping series.

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \ldots = \frac{1}{2} \cdot \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}}\right] + \frac{1}{2}\left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right] + \frac{1}{2}\left[\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right] + \cdots + \frac{1}{2}\left[\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right]$$

pts: /15

$$= \text{after cancellation} = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1}$$

$$\therefore \text{ sum of series} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1} = \boxed{\frac{1}{2}}$$

3. (5 pts each) Determine whether the following series converge or diverge. Give reasons for your answers.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$;

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$\therefore$ diverges because of the n-th term divergence te

(b) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$; use the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \frac{(2n+1)}{n!}$$

all terms are positive

$$= \lim_{n\to\infty} \frac{\cancel{n!}(n+1)\ \cancel{(2n+1)!}}{(2n+3)\underbrace{\cancel{(2n+2)}}_{2}\cancel{(2n+1)!}\ \cancel{n!}} = \lim_{n\to\infty} \frac{1}{2(2n+3)} = 0 < 1$$

$\therefore$ Converges

(c) $\sum_{n=1}^{\infty} 1+(-1)^n$.

$a_1 = 0$

$a_2 = 2$

$a_3 = 0$

$a_4 = 2$

$\vdots$

$\therefore \lim_{n\to\infty} a_n$ does not exist

$\therefore$ series diverges

pts: /15

4. Determine whether the following series converge absolutely, converge conditionally, or diverge. Give reasons for your answers.

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$;

the series $\boxed{\text{converges absolutely}}$

Infact $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2+2n+1}$ and this series converges because of the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (a converging p-series $p=2>1$)

(*b*) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$.

It does not converge absolutely

In fact $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

which diverges because it is a p-series with $p = \frac{1}{2} < 1$!!

But it converges. So it converges conditionally

In fact we need to check that

$a_n = \frac{1}{\sqrt{n}} \geq 0$ ✓

$a_n \to 0$ as $n \to \infty$ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

$a_{n+1} \leq a_n$ for all n (decreasing)

$n < n+1 \rightsquigarrow \sqrt{n} < \sqrt{n+1} \rightsquigarrow$

$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$ ✓

pts: /15

5. Determine whether the following series converges or not:

$$\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)} = \sum_{n=1}^{\infty} a_n$$

Will it be of any help if you know the behaviour of the improper integral

$$\int_1^{\infty} \frac{dx}{x(1+\ln^2 x)} \, ? = \int_1^{\infty} f(x)\,dx$$

Explain....and compute.

Notice $a_n = f(n)$ for all n. Moreover $f(x)$ is a decreasing function. In fact

$$f'(x) = \frac{-(1+\ln^2 x) - x \cdot 2\ln x \cdot \frac{1}{x}}{x^2(1+\ln^2 x)^2} = \frac{-1-\ln^2 x - 2\ln x}{x^2(1+\ln^2 x)^2} < 0$$

Hence the convergence of the series is the same as the one of the integral.

$$\int_1^{\infty} \frac{dx}{x(1+\ln^2 x)} = \lim_{b\to\infty} \int_1^{b} \frac{du}{1+u^2} = \lim_{b\to\infty} \left[\tan^{-1} u\right]_1^b = \lim_{b\to\infty} \left[\tan^{-1} b - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < +\infty$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\therefore$ Converges by integral test

pts: /10

6. Determine whether the following statements are true (**T**) or false (**F**). Check the appropriate box.

T	F	
☐	☑	If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is certainly convergent.
☑	☐	If $\lim_{n\to\infty} a_n = 1/2$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
☐	☑	The series $\sum_{n=1}^{\infty} 3^n$ is convergent.
☐	☑	The series $\sum_{n=1}^{\infty} 3^{-n}$ is divergent.
☐	☑	If a series converges then it converges absolutely.

pts: /10

7. (*a*) (5 pts) Find the interval of convergence of the following power series

$$\sum_{n=1}^{\infty} \sqrt[n]{n}(x-5)^n.$$

Recall that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

use the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[n+1]{n+1}\,(x-5)^{n+1}}{\sqrt[n]{n}\,(x-5)^n}\right| =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\cdot|x-5| = \frac{1}{1}\cdot|x-5| < 1$$

$\therefore\ |x-5|<1$ $\quad \boxed{4 < x < 6}$ But not at the end points

(*b*) (10 pts) Find the series' interval of convergence and, within this interval, the sum $f(x)$ of the series

$$\sum_{n=1}^{\infty}\frac{(x+1)^{2n}}{9^n} = \underline{\qquad\qquad\qquad}.$$

Observe that this is a geometric series -

$$= \sum_{n=1}^{\infty}\left(\frac{(x+1)^2}{9}\right)^n = \frac{(x+1)^2}{9} + \left(\frac{(x+1)^2}{9}\right)^2 + \left(\frac{(x+1)^2}{9}\right)^3 + \cdots$$

$$= \frac{(x+1)^2}{9}\left[1 + \frac{(x+1)^2}{9} + \left(\frac{(x+1)^2}{9}\right)^2 + \cdots\right]$$

$$= \frac{(x+1)^2}{9}\cdot\frac{1}{1-\frac{(x+1)^2}{9}} = \frac{(x+1)^2}{9}\cdot\frac{9}{9-(x+1)^2} = \frac{(x+1)^2}{9-(x+1)^2}$$

Convergence for $\left|\frac{(x+1)^2}{9}\right| < 1$ or $\left|\frac{x+1}{3}\right| < 1$

But not at the end points

$\Longrightarrow$ $|x+1| < 3$ $\quad -4 < x < 2$

pts: /15

8. Find a power series representation for the function $f(x) = \ln(1+x)$ and determine the radius of convergence.

We have seen this in class: geom series

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \frac{1}{1-(-t)}\,dt =$$

$$= \int_0^x (1 - t + t^2 - t^3 + t^4 \ldots)\,dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$\boxed{\text{term by term integration}}$

which conv. for $-1 < x \leq 1$

It is a homework in the book ~~to~~ show that that it converges for $x = 1$ to $\ln 2$ (it is the alternating harmonic series)

Obviously, no convergence for $x = -1$.

pts: /10

Bonus. Use series to evaluate the limit

$$\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x} = \boxed{-1}.$$

Taylor series of

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\therefore \boxed{1 + x - e^x = -\frac{x^2}{2!} - \frac{x^3}{3!} - \ldots}$$

$$= \lim_{x \to 0} \frac{x^2/2! - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots}{-\frac{x^2}{2!} - \frac{x^3}{3!} - \ldots}$$

$$= \lim_{x \to 0} \frac{1/2! - \frac{x^2}{4!} + \frac{x^4}{6!}}{-1/2! - \frac{x}{3!} - \ldots}$$

$$= \frac{1/2!}{-1/2!} = \boxed{-1}$$

Why not check up all your work?

pts: /5

$$\boxed{1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots}$$