Ma123: Elementary Calculus and Its Applications
Exam II, October 22, 2003

Instructions: There are 9 problems on 7 pages (including this cover page). Each problem is worth 10 points. Check that you have a complete exam.

Show all work and explain your answers. Unsupported answers will receive no credit. Fill in the information below and put your name or initials on each of the other pages.

Name: ________________________________

Instructor: ________________________________

Section Number/Class Meeting Time: ________________

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Problem 1: Calculate the derivative of each of the following. 
You do not need to simplify your answer.

a. \((2x^3 + 1)^5\)
   \[
f'(x) = 5(2x^3 + 1)^4 \cdot (6x^2)
   \]

b. \(\frac{x}{1-\sqrt{2}x} = \frac{x}{1-\sqrt{2}\sqrt{x}}\)
   [recall]
   \[
   \int \frac{1}{\sqrt{x}} \, dx = \frac{1}{2\sqrt{x}}
   \]
   use quotient rule and this fact
   \[
f'(x) = \frac{1 \cdot (1 - \sqrt{2}\sqrt{x}) - x \cdot (-\sqrt{2} \cdot \frac{1}{2\sqrt{x}})}{(1 - \sqrt{2}\sqrt{x})^2}
   \]

Problem 2: Suppose a particle moves along the x-axis to that its position at time \(t\) is \(x(t) = -3t^2 - 21t + 90\) feet.

a. What is the average velocity of the particle over the time interval \(3 \leq t \leq 6\)?
   \[
   \text{ave. vel.} = \frac{x(6) - x(3)}{6 - 3} = \frac{-144 - 0}{3} = -\frac{144}{3} = -48 \text{ feet/sec}
   \]

b. Which direction (left or right) is the particle moving at time \(t = 0\)? Explain your answer.
   \(v'(t) = -6t - 21\) , \(v'(0) = -21\) i.e. it moves left

c. At what time(s) is the particle stationary? That is, at what times does it stop moving?
   \[
v'(t) = 0 \iff -6t - 21 = 0
   \]
   \[
   \therefore t = - \frac{21}{6} = -\frac{7}{2} \text{ sec}
   \]
Problem 3: The diagram is a sketch of a portion of the graph of \( f(x) = -\frac{x^3}{3} + 3x^2 - 5x + 1 \)

\[ f'(x) = -x^2 + 6x - 5 = 0 \]
\[ \iff x^2 - 6x + 5 = 0 \iff (x-5)(x-1) = 0 \iff x = 1, 5 \]

\( f(x) \text{ increases on } (1, 5) \)

b. Calculate all intervals on which the graph of \( f(x) \) is concave down.

\[ f''(x) = -2x + 6 = 0 \iff x = 3 \]

\[ \text{sign } f'' = \begin{cases} + & x < 3 \\ - & x > 3 \end{cases} \]

\( f(x) \text{ concave down on } (3, +\infty) \)

b. Calculate the absolute maximum value of the function \( f(x) \) on the interval \(-1 \leq x \leq 7\)

Need to check \( f(-1) = \frac{9}{3} = 3 \) and the value at the end points \( f(7) = -1.33 \) and the critical point \( f(5) = \frac{9}{3} = 3 \)

Problem 4:

a. Find a numbers \( A \) such that \( f(x) = Ax^3 + 3x + 7 \) has a critical point at \( x = 2 \).

\[ f'(x) = 3Ax^2 + 3 \quad \text{want } f'(2) = 0 \iff 3A \cdot 2^2 + 3 = 0 \]

\[ \therefore A = -\frac{1}{4} \]

b. Find numbers \( C \) and \( D \) such that the graph of \( f(x) = Cx^3 + 3x^2 + D \) has an inflection point at \((-1, 7)\)

\[ f'(x) = 3Cx^2 + 6x, \quad f''(x) = 6Cx + 6 \]

\[ \text{want } f''(-1) = 0 \iff -6C + 6 = 0 \iff C = 1 \]

Also \((-1, 7)\) is on the graph of \( f \), so

\[ f(-1) = 7 \iff f(-1) = \frac{1}{C}(-1)^3 + 3(-1)^2 + D = 7 \]

\[ \therefore D = 5 \]
Problem 5: Sketch the graph of a function with all of the following properties:

a. \( f(x) \) is defined only for \( x \) such that \( 0 \leq x \leq 9 \)
b. \( (0,0) \) and \( (4,3) \) are on the graph of \( f(x) \)
c. \( 0 < f'(x) \) for all \( x \) such that \( 0 \leq x \)
d. \( f''(x) < 0 \) for all \( x \) such that \( x < 4 \)
e. \( 0 < f''(x) \) for all \( x \) such that \( x > 4 \)

Problem 6: Suppose \( f(x) \) is a function such that \( f(3) = -1 \) and \( f'(3) = 5 \)

a. What is the equation of the tangent line to the graph of \( h(x) = f(x)^2 \) at \( x = 3 \)?
\[
h(3) = (f(3))^2 = (-1)^2 = 1 \quad \therefore \quad P(3,1) \quad \therefore \quad h'(3) = 2f(3)f'(3) = 2(-1)(5) = -10
\]
b. Is the function \( p(x) = (7 - f(x))^2 \) increasing at \( x = 3 \)? Justify your answer. The justification is what will be graded.
\[
p'(x) = 2(7 - f(x))(7 - f'(x))
p'(3) = 2(7 - 1)(7 - 5) = 88 > 0 \quad \therefore \quad \text{the function is increasing at } x = 3
\]

\[
y - 1 = -10(x - 3) \\
\therefore \quad y = -10x + 31
\]
Problem 7. The graph is given of the derivative of the function \( f(x) \). That is, this is the graph of \( f'(x) \), not \( f(x) \).

a. At which of the points \{A, B, C, D, E, F, G, H, I\} is \( f(x) \) increasing? (Note this asks about \( f(x) \), the graph is of \( f'(x) \).)

\[ A, E, F, G, H \]

b. At which of the points \{A, B, C, D, E, F, G, H, I\} does \( f(x) \) have a local maximum of \( f(x) \)? (This is not asking for local maxima of \( f'(x) \)).

\[ B, I \]

c. At which of the points \{A, B, C, D, E, F, G, H, I\} does \( f(x) \) have a local minimum? (Note this asks about \( f(x) \), the graph is of \( f'(x) \)).

\[ D \]

d. For \( A \leq x \leq I \) describe the intervals (using choices from A through I as end points) on which \( f(x) \) is concave down.

Note: you only need to give the largest intervals. For instance if \( f(x) \) were increasing on \( (P, Q) \) and \( (P, R) \) with \( Q < R \) then you would only need to give \( (Q, R) \).

\[ (A, C) \quad \text{and} \quad (G, I) \]
8. Two roads meet at an angle of 90 degrees, one running North-South and the other running East-West.

Car A and Car B are both at the intersection at time \( t = 0 \) hours.
Car A travels north of the intersection at 40 miles per hour.
Car B is east of the intersection traveling 30 miles per hour.

\[ v_A = \frac{dy}{dt} = 40 \text{ mph} \]
\[ v_B = \frac{dx}{dt} = 30 \text{ mph} \]

when A is at 4 miles
and B is at 3 miles

\[ z = \text{dist}(A, B) = \sqrt{3^2 + 4^2} = 5 \]

At what rate (correct to .001) in miles per hour is the distance between the cars changing when car A is 4 miles from the intersection and car B is 3 miles from the intersection?

\[ z(t)^2 = x(t)^2 + y(t)^2 \]

\[ 2z(t) \frac{dz}{dt} = 2x(t) \frac{dx}{dt} + 2y(t) \frac{dy}{dt} \]

\[ \frac{dz}{dt} = \frac{x(t) \frac{dx}{dt} + y(t) \frac{dy}{dt}}{z(t)} \]

we have \[ \frac{3 \cdot 30 + 4 \cdot 40}{5} = \frac{50}{5} = 50 \text{ mph} \]
Problem 9: An open rectangular box is to be made from a 16 by 16-inch piece of cardboard by cutting a squares out of the corners and folding up the flaps (see the diagram). Find the value of \( x \) for which the volume of the box will be a maximum.

Volume of box

\[
\text{area base} \cdot \text{height} = (16-2x)^2 \cdot x
\]

\[
= 4(8-x)^2 \cdot x
\]

defined on \( 0 \leq x \leq 8 \)

The maximum can occur at the end points of the interval \( x = 0 \), \( x = 8 \) or at the critical values.

\[
V'(x) = 4 \cdot 2(8-x)(-1) \cdot x + 4(8-x)^2
\]

Use product rule and generalized power rule.

\[
V'(x) = 4(8-x)[-2x + (8-x)] = 4(8-x)(8-3x)
\]

Critical numbers are \( x = 8 \), \( x = \frac{8}{3} \)

Check: \( V(0) = V(8) = 0 \) \( V\left(\frac{8}{3}\right) = 308.407 \text{ in.}^3 \)