1. (a) $\int_C f(x,y,z)\,ds$ when $f(x,y,z)=\sqrt{y+x+4z}$

and $C=(2t+8)\hat{\imath}+(-2t)\hat{\jmath}+t^2\hat{k}$ $\quad 0\le t\le 1.$

$$=\int_0^1 f(x(t),y(t),z(t))\,|\vec{v}(t)|\,dt =$$

$$=\int_0^1 \sqrt{(-2t)+(2t+8)+4t^2}\cdot\sqrt{2^2+(-2)^2+(2t)^2}\,dt$$

$$=\int_0^1 \sqrt{8+4t^2}\cdot\sqrt{8+4t^2}\,dt=\int_0^1 (8+4t^2)\,dt$$

$$=8t+\frac{4}{3}t^3\Big]_0^1 = 8+\frac{4}{3}=\boxed{\frac{28}{3}}$$

1. (b) $\int_C \vec{F}\cdot\hat{T}\,ds=\int_C \vec{F}\cdot d\vec{r} =$

$$=\int_0^1\left((2t+8-2t)\hat{\imath}+((-2t)t^2)\hat{\jmath}+(-2t)^2\hat{k}\right)\cdot$$

$$\cdot\left(2\hat{\imath}-2\hat{\jmath}+2t\hat{k}\right)dt$$

$$=\int_0^1 16+(+4t^3)+(8t^3)\,dt$$

$$=\int_0^1 (16+12t^3)\,dt = 16t+3t^4\Big]_0^1 = \underline{19}$$

2. (a) $\vec{F} = 2xz\,\hat{i} + z\,\hat{j} + (x^2+y)\,\hat{k}$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

or $\frac{\partial f}{\partial x} = 2xz \qquad \frac{\partial f}{\partial y} = z \qquad \frac{\partial f}{\partial z} = x^2+y$.

$\to f(x,y,z) = x^2 z + g(y,z)$

Thus $\frac{\partial f}{\partial y} = 0 + \frac{\partial}{\partial y} g(y,z) \overset{\text{want}}{=} z$

$\therefore\ g(y,z) = yz + h(z)$

$\therefore f(x,y,z) = x^2 z + yz + h(z)$

Finally $\frac{\partial f}{\partial z} = \cancel{x^2+y} + h'(z) = \cancel{x^2+y}$

$\therefore\ h'(z) = 0 \implies h(z) = \text{constant}.$

$\therefore f(x,y,z) = \underline{x^2 z + yz + \text{const}}$

(b) $\int_C \vec{F}\cdot d\vec{r} = f(1,2,3) - f(0,1,1)$

$= (1^2\cdot 3 + 2\cdot 3 + \cancel{\text{const}}) - (0^2\cdot 1 + 1\cdot 1 + \cancel{\text{const}})$

$= 9 - 1 = \underline{8}$

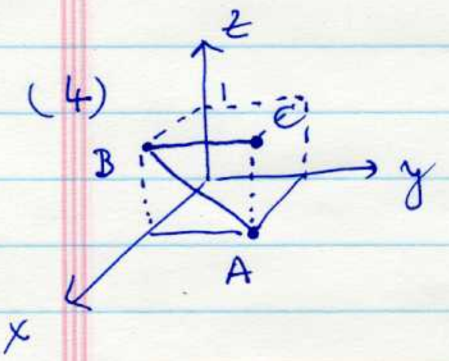
(3) $\vec{F} = xy^2\,\hat{\imath} + 3x\,\hat{\jmath} = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (3 - 2xy)\,dx\,dy$$

$$= \int_0^1 \left(\int_0^1 (3-2xy)\,dy\right) dx = \int_0^1 \Big[3y - xy^2\Big]_0^1 dx$$

$$= \int_0^1 (3-x)\,dx = 3x - \tfrac{1}{2}x^2\Big]_0^1 = 3 - \tfrac{1}{2} = \frac{5}{2}$$

(4)



Segment from A to B

$$\vec{r}_1(t) = (1+0t)\,\hat{\imath} + (1-t)\,\hat{\jmath} + (0+t)\,\hat{k}$$

$$= \hat{\imath} + (1-t)\,\hat{\jmath} + t\hat{k} \qquad 0 \le t \le 1$$

Segment from B to C

$$\vec{r}_2(t) = (1+0t)\,\hat{\imath} + (0+t)\,\hat{\jmath} + (1+0t)\,\hat{k} =$$

$$= \hat{\imath} + t\,\hat{\jmath} + \hat{k} \qquad 0 \le t \le 1$$

Thus $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_0^1 \left(1\cdot(1-t)\,\hat{\imath} + (1-t)t\,\hat{\jmath} + 1\cdot t\,\hat{k}\right)\cdot$

$$\cdot\,(0\,\hat{\imath} - \hat{\jmath} + \hat{k})\,dt$$

$$= \int_0^1 (t^2 - t) + t\,dt = \int_0^1 t^2\,dt = \tfrac{1}{3}t^3\Big]_0^1 = \tfrac{1}{3}$$

Whereas $\int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_0^1 \left(1 \cdot t\hat{i} + t \cdot 1\hat{j} \;\; 1 + \hat{k} \right) \cdot$

$\cdot \left(0\hat{i} + \hat{j} + 0\hat{k} \right) dt$

$= \int_0^1 t \, dt = \frac{1}{2} t^2 \Big]_0^1 = \frac{1}{2}$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{1}{3} + \frac{1}{2} = \underline{\frac{5}{6}}$

(5) $\int_C \sqrt{x^2+y^2} \, ds$ when C is given by

$\vec{r}(t) = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 3t\,\hat{k}$

$-2\pi \leq t \leq 2\pi$

$= \int_{-2\pi}^{2\pi} \sqrt{(4\cos t)^2 + (4\sin t)^2} \cdot \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2} \, dt$

$= \int_{-2\pi}^{2\pi} 4 \cdot \sqrt{16 + 9} \, dt = \int_{-2\pi}^{2\pi} 4 \cdot 5 \, dt =$

$= 20 \int_{-2\pi}^{2\pi} dt = 20 \cdot 4\pi = \underline{100\pi}$

(6) (a) Verify that a potential function for

$$\vec{F}(x,y,z) = 2x\,\hat{i} + (2y+z)\,\hat{j} + (y+1)\,\hat{k}$$

is given by $f(x,y,z) = x^2 + y^2 + yz + z + \text{Const}$

$\therefore \vec{F} = \vec{\nabla} f$.

(7) $$\int_{(1,1,1)}^{(0,3,2)} y e^x\,dx + e^x\,dy + 2z\,dz = \ldots$$

Since no path is specific, it is almost obvious that the vector field better be conservative.

In fact $\vec{F} = y e^x\,\hat{i} + e^x\,\hat{j} + 2z\,\hat{k}$

has the following potential function:

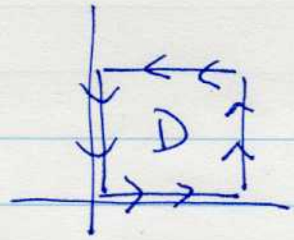
$$f(x,y,z) = y e^x + z^2 + \text{Const}$$

$$\therefore \int_{(1,1,1)}^{(0,3,2)} \vec{F}\cdot d\vec{r} = y e^x + z^2 + \text{const}\Big]_{(1,1,1)}^{(0,3,2)}$$

$$= \left(3e^0 + 2^2 + \cancel{\text{const}}\right) - \left(1\cdot e^1 + 1 + \cancel{\text{const}}\right)$$

$$= 3 + 4 - e - 1 = \boxed{6 - e}$$

(8) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$



$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy = \iint_D 1 - (-1)\, dx\,dy =$$

$$= \iint_D 2\, dx\,dy = 2 \text{ area of } D = 2$$

(9) (a) the segment from $(0,0)$ to $(2,4)$

has parametrization

$$\vec{r}(t) = (0+2t)\,\hat{\imath} + (0+4t)\,\hat{\jmath} \quad 0 \le t \le 1$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(2t(4t)^2 + e^{6t}\right)\hat{\imath} + \left((2t)^2 4t + 4t + \sin(4t)\right)\hat{\jmath}$$

$$\cdot\,(2\hat{\imath} + 4\hat{\jmath})\, dt$$

$$= \int_0^1 \left(64t^3 + 2e^{6t} + 64t^3 + 16t + 4\sin 4t\right) dt$$

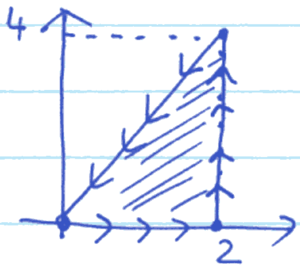
$$= \int_0^1 \left[128t^3 + 16t + 2e^{6t} + 4\sin(4t)\right] dt$$

$$= 32t^4 + 8t^2 + \frac{1}{3}e^{6t} - \cos(4t)\Big]_0^1 =$$

$$= \left[32 + 8 + \tfrac{1}{3}e^{6} - \cos(4)\right] - \left[\tfrac{1}{3} - \overset{=1}{\cos(0)}\right]$$

$$= \ldots$$

(b)



$$\int_C \vec{F} \cdot d\vec{r} =$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_D \big((2xy + 2) - 2xy\big)\, dx\,dy = \iint_D 2\, dx\,dy =$$

$$= 2\,\text{area}(D) = 2 \cdot \frac{2 \cdot 4}{2} = \underline{8}$$

(10) (a)

$$\frac{\partial f}{\partial x} = \cos y - x$$

$$\frac{\partial f}{\partial y} = z e^{yz} - x \sin y$$

$$\frac{\partial f}{\partial z} = y e^{yz}$$

$$\frac{\partial f}{\partial x} = \cos y - x \longrightarrow f = x\cos y - \frac{1}{2}x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = -\cancel{x \sin y} + \frac{\partial}{\partial y} g(y,z) \overset{\text{WANT}}{=} z e^{yz} - \cancel{x \sin y}$$

$$\therefore \frac{\partial g}{\partial y} = z e^{yz} \longrightarrow g(y,z) = e^{yz} + h(z)$$

$$\therefore f = x\cos y - \frac{1}{2}x^2 + e^{yz} + h(z)$$

Finally $\frac{\partial f}{\partial z} = y e^{yz} + h'(z) \overset{\text{WANT}}{=} y e^{yz}$

$$\longrightarrow h'(z) = 0 \longrightarrow h(z) = \text{const}.$$

$$\therefore \underline{f(x,y,z) = x\cos y - \frac{1}{2}x^2 + e^{yz} + \text{const}}$$

(b) $\displaystyle\int_{(0,0,1)}^{(1,\pi,0)} \vec{F}\cdot d\vec{r} = f(1,\pi,0) - f(0,0,1) =$

$$= \left(1\cdot\cos\pi - \frac{1}{2} + \cancel{e^0} + \cancel{\text{Const}}\right) - \left(\cancel{e^0} + \cancel{\text{Const}}\right)$$

$$= -1 - \tfrac{1}{2} = -\frac{3}{2}$$

(11) $\displaystyle\oint (xy^2 + 2y + 5)\,dx + (y^3\ln y + x^2y)\,dy$

around $x^2+y^2=9$

$$= \iint_{\text{Disk}} \left[(2xy) - (2xy+2)\right] dx\,dy =$$

$$= \iint_{\text{Disk}} -2\,dx\,dy = -2\,\text{area}(\text{Disk}) = -2\pi\cdot 3^2$$

$$= \underline{-18\pi}$$

(12) $\vec{F} = (y + 2xz)\,\hat{i} + x\,\hat{j} + x^2\,\hat{k}$

Observe that $\vec{F} = \vec{\nabla} f$ when

$f = xy + x^2 z + \text{Const}$

Thus $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = f(1,1,1) - f(0,0,0)$

$= \left[1\cdot 1 + 1^2\cdot 1 + \text{const}\right] - \left[0\cdot 0 + 0^2\cdot 0 + \text{const}\right]$

$= 2$

(13) $\vec{F} = (2xy + e^{z-x})\,\hat{i} + (x^2 - \sin y)\,\hat{j} - e^{z-x}\,\hat{k}$

(a) ....

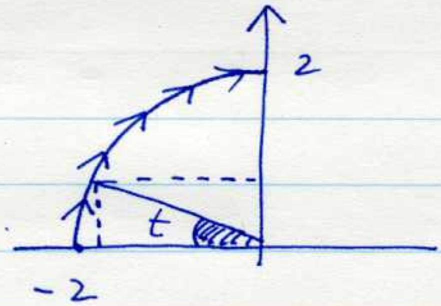
$\vec{F} = \vec{\nabla} f$ when $f = \underline{x^2 y + \cos y - e^{z-x} + \text{const}}$

(b) $\displaystyle\int_{(0,0,0)}^{(1,\pi,1)} \vec{F}\cdot d\vec{r} = f(1,\pi,1) - f(0,0,0) =$

$= (1^2\cdot\pi + \cos\pi - e^0 + \cancel{\text{const}}) - (0^2\cdot 0 + \cos 0 - e^0 + \cancel{\text{const}})$

$= \pi - 1 - \cancel{1} - 1 + \cancel{1} = \underline{\pi - 2}$

(14)



Observe that a parametrization for that arc of circumference is:

$$\vec{r}(t) = (\overbrace{-2\cos t}^{x(t)})\hat{i} + (\overbrace{+2\sin t}^{y(t)})\hat{j}$$

$$0 \le t \le \frac{\pi}{2}$$

$$\therefore \int_C xy\,dx + dy = \int_0^{\pi/2} \underbrace{(-2\cos t)}_{x}\underbrace{(2\sin t)}_{y}\,\underbrace{2\sin t\,dt}_{dx} +$$

$$+\ 2\cos t\,dt =$$

$$= \int_0^{\pi/2} \left(-8\sin^2 t\cos t + 2\cos t\right) dt =$$

$$= -\frac{8}{3}\sin^3 t + 2\sin t \Big]_0^{\pi/2} = -\frac{8}{3} + 2 = \underline{\frac{-2}{3}}$$