Practice Problems (Chapter 14)

1. Let $C$ be the curve in space parametrized by

$$\vec{r}(t) = (2t+8)\hat{i} + (-2t)\hat{j} + t^2\hat{k} \quad 0 \leq t \leq 1$$

(a) Evaluate the line integral

$$\int_C f(x,y,z) \, ds$$

where $f(x,y,z) = \sqrt{y + x + 4z}$

(b) Evaluate the work $\int_C \vec{F} \cdot d\vec{s}$, where

$$\vec{F}(x,y,z) = (x+y)\hat{i} + (yz)\hat{j} + y^2\hat{k}$$

and $C$ is as above.

2. Let $\vec{F} = 2xz\hat{i} + z\hat{j} + (x^2+y)\hat{k}$ be a 3-dimensional vector field.

(a) Show that $\vec{F}$ is conservative.

(b) Find the work done by $\vec{F}$ over any path from $(0,1,1)$ to $(1,2,3)$.

3. Let $\vec{F} = xy^2\hat{i} + 3x\hat{j}$ be a 2-dimensional vector field. Use Green's theorem to calculate the counterclockwise circulation of $\vec{F}$ around the unit square, as shown.

\[\begin{array}{c}
(0,0) \\
(0,1) \\
(1,1) \\
(1,0)
\end{array}\]
4. Find the work done by the force \[ \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k} \]

over the path consisting of the line segment from the point \((1,1,0)\) to the point \((1,0,1)\) followed by the line segment from \((1,0,1)\) to \((1,1,1)\).

5. Evaluate \( \int_C \sqrt{x^2+y^2} \, ds \) along the curve \( \mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3t \mathbf{k} \) where \(-2\pi \leq t \leq 2\pi\).

6. Consider the vector field \[ \mathbf{F}(x, y, z) = 2x \mathbf{i} + (2y+z) \mathbf{j} + (y+1) \mathbf{k} \]

(a) Show that \( \mathbf{F} \) is a conservative vector field.

(b) Find a potential function \( f \) for the field \( \mathbf{F} \).

7. Evaluate \( \int_C y e^x \, dx + e^x \, dy + 2z \, dz \) where \( (1,1,1) \).
8. Use Green's theorem to find the work done by
\[ \vec{F}(x, y) = (3x - y) \hat{i} + (x - 2y) \hat{j} \]
in moving a particle once counterclockwise around the space \( C \), where \( C \) is the boundary of the square \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \).

9. Consider the line integral
\[ (*) \quad \oint_C (xy^2 + e^{3x}) \, dx + (x^2 y + 2x + \sin y) \, dy. \]

(a) Compute \((*)\) along the segment from \((0,0)\) to \((2,4)\).

(b) If \( C \) is the counterclockwise boundary of the triangular region with vertices \((0,0)\), \((2,0)\), and \((2,4)\), use Green's Theorem to evaluate \((*)\).

10. Consider the vector field
\[ \vec{F}(x, y, z) = (\cos y - x) \hat{i} + (ze^{yz} - x \sin y) \hat{j} + ye^{yz} \hat{k}. \]

(a) Assuming \( \vec{F} = \nabla f \), find the potential \( f \).
(b) Use \( f \) calculated in part (a) to evaluate
\[
\int_{(0, 0, 1)}^{(1, \pi, 0)} \mathbf{F} \cdot d\mathbf{r}
\]

11. Evaluate the line integral using Green's Theorem
\[
\oint (xy^2 + 2y + 5)\,dx + (y^3 \sin y + x^2 y)\,dy
\]
in counterclockwise direction around the circle
\[x^2 + y^2 = 9.\]

12. Find the work done by the force
\[
\mathbf{F}(x, y, z) = (y + 2xz)\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}
\]
over the straight line path from \((0, 0, 0)\) to \((1, 1, 1)\).

13. Consider the vector field
\[
\mathbf{F}(x, y, z) = (2xy + e^{2-x})\,\mathbf{i} + (x^2 - \sin y)\,\mathbf{j} - e^{2-x}\,\mathbf{k}
\]
(a) Find a potential function for \( \mathbf{F}(x, y, z) \).
(b) Find the exact value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \).
along the straight path from $(0,0,0)$ to $(1, \pi, 1)$.

14. Let $C$ be the arc of the circle of radius 2 as shown in the figure.

Find \[ \int_C xy \, dx + dy. \]