Review Questions for Math 265 Final

The final examination will be a combination of multiple choice, short answer, and standard problems. The samples below are designed to illustrate possible styles of the questions. These samples may not include all topics that might be covered on the actual final.

SAMPLE MULTIPLE CHOICE QUESTIONS: From a recent 262 final.

1. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ 1 & 3 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a & \frac{1}{3} & \frac{1}{3} \\ 1 & b & 0 \\ -2 & -\frac{1}{3} & c \end{bmatrix}$, then

   A. $a = 0, b = 1, c = 0$ B. $a = 0, b = -1\frac{1}{6}, c = -\frac{1}{3}$ C. $a = -1, b = 0, c = -\frac{1}{3}$
   D. $a = 0, b = 0, c = \frac{2}{3}$ E. $a = -1, b = 0, c = \frac{2}{3}$

2. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 9$, calculate $\det \begin{pmatrix} g+2a & h+2b & i+2c \\ a & b & c \\ 3d & 3e & 3f \end{pmatrix}$

   A. 3 B. -3 C. 27 D. -27 E. -54

3. Which of the following is a subspace of $\mathbb{R}^2$?

   A. The set of all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 \geq 0, x_2 \geq 0$.

   B. The set of all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 - x_2^2 = 0$.

   C. The set of all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 \cdot x_2 = 1$.

   D. The set of all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 + x_2 = 1$.

   E. The set of all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 + 2x_2 = 0$.

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. Then, a basis for $Null(T)$ is

   A. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ C. $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

5. Suppose an $n \times n$ matrix B is obtained from A by a sequence of row operations. Which of the following statements is <u>not</u> always true?

   A. rank B = rank A
   B. The nullspaces $Null(T)$ and $Null(S)$ of the linear transformations $T(\mathbf{x}) = A\mathbf{x}$ and $S(\mathbf{x}) = B\mathbf{x}$ are the same
   C. A and B have the same eigenvalues
   D. If A has an inverse then B has an inverse
   E. If $\det A = 0$ then $\det B = 0$

6. The dimension of the space spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$ is

   A. 0 B. 1 C. 2 D. 3 E. 4

7. Let $M_2(\mathbb{R})$ be the vector space of all 2×2 matrices with matrix addition and multiplication by a scalar. The dimension of the subspace W spanned by $\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ is

A. 0 B. 1 C. 2 D. 3 E. 4

8. Suppose that A is an $n \times n$ matrix, $\mathbf{b}$ is an n component column vector, and $A\mathbf{x} = \mathbf{b}$ has no solution. Then

A. $\text{rank}(A) = n$ B. A^{-1} exists C. $\det(A) \neq 0$ D. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution E. None of the above

9. If $A = \begin{pmatrix}1 & 1\\ -5 & 1\end{pmatrix}$ then the eigenvalues of A are

A. -1 and 4 B. -1 and -1 C. -2 and 3
D. 1 and 2 E. $2i$ and $-2i$

10. The matrix $A = \begin{pmatrix}1 & -1\\ 2 & 4\end{pmatrix}$ has eigenvalues 2 and 3. An eigenvector corresponding to 2 is

A. $\begin{bmatrix}1\\1\end{bmatrix}$ B. $\begin{bmatrix}-1\\1\end{bmatrix}$ C. $\begin{bmatrix}1\\2\end{bmatrix}$ D. $\begin{bmatrix}-1\\4\end{bmatrix}$ E. $\begin{bmatrix}0\\0\end{bmatrix}$

11. The matrix $A = \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ -2 & -2 & -1\end{pmatrix}$ has $\lambda = 1$ as an eigenvalue. What is the dimension of the eigenspace corresponding to $\lambda = 1$?

A. 0 B. 1 C. 2 D. 3 E. 4

12. A basis of the solution space of $\frac{d\mathbf{x}}{dt} = \begin{pmatrix}-3 & -1\\ 2 & -1\end{pmatrix}\mathbf{x}$

A. $\mathbf{x}_1 = e^{-2t}\cos t\begin{pmatrix}-1\\1\end{pmatrix} - e^{-2t}\sin t\begin{pmatrix}0\\1\end{pmatrix}$ $\mathbf{x}_2 = e^{-2t}\cos t\begin{pmatrix}-2\\2\end{pmatrix} - e^{-2t}\sin t\begin{pmatrix}0\\2\end{pmatrix}$

B. $\mathbf{x}_1 = e^{-t}\cos 2t\begin{pmatrix}-1\\1\end{pmatrix} - e^{-t}\sin 2t\begin{pmatrix}0\\1\end{pmatrix}$ $\mathbf{x}_2 = e^{-t}\sin 2t\begin{pmatrix}-2\\2\end{pmatrix} + e^{-t}\cos 2t\begin{pmatrix}0\\2\end{pmatrix}$

C. $\mathbf{x}_1 = e^{-t}\cos 2t\begin{pmatrix}-1\\1\end{pmatrix} - e^{-t}\sin 2t\begin{pmatrix}0\\1\end{pmatrix}$ $\mathbf{x}_2 = e^{-t}\sin 2t\begin{pmatrix}-1\\1\end{pmatrix} + e^{-t}\cos 2t\begin{pmatrix}0\\1\end{pmatrix}$

D. $\mathbf{x}_1 = e^{-2t}\cos t\begin{pmatrix}-1\\1\end{pmatrix} - e^{-2t}\sin t\begin{pmatrix}0\\1\end{pmatrix}$ $\mathbf{x}_2 = e^{-2t}\sin t\begin{pmatrix}-1\\1\end{pmatrix} + e^{-2t}\cos t\begin{pmatrix}0\\1\end{pmatrix}$

E. $\mathbf{x}_1 = e^{-t}\cos t\begin{pmatrix}-2\\2\end{pmatrix} - e^{-t}\sin t\begin{pmatrix}0\\2\end{pmatrix}$ $\mathbf{x}_2 = e^{-t}\sin t\begin{pmatrix}-2\\2\end{pmatrix} + e^{-t}\cos t\begin{pmatrix}0\\2\end{pmatrix}$

SAMPLE problems after 6.4 (6.5, 7.1, 5.2,5.3)

1. Let W be the plane spanned by the vectors $(1, -1, 1, 1)$ and $(1, 1, -1, 1)$ in $\mathbb{R}^4$.

   a) Find the point $\vec{b_0}$ on W closest to $\vec{b} = (1, 2, 3, 4)$.

   b) Find the distance $||\vec{b_0} - \vec{b}||$.

c) Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Show that $A\vec{x} = \vec{b}$ is inconsistent. Give the least mean squares solution to $A\vec{x} = \vec{b}$.

2. Let B be a real symmetric 3×3 matrix with eigenvalues $\{2, -2, 4\}$ and eigenvectors $(-1, 0, 1)$, $(1, 1, 1)$, $(1, -2, 1)$.

   a) What are the eigenvalues of B^{99} ?

   b) Find a matrix V such that

$$V^{-1}BV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D.$$

   c) What is B ?

3. a) Let A be an $n \times n$ complex matrix such that $A' = A$. Suppose $\vec{v}$ is non-zero. Show that if $A\vec{v} = \lambda\vec{v}$, then λ is a real number.

   b) Suppose U is a unitary matrix. Show that $|det(U)|$ is one.

   c) Given an example of an orthogonal matrix K with determinant -1.

4. In $\mathbb{R}^3$, the plane W is the null space of $N = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$.

   a) Find an orthonormal basis of W.

   b) Give a matrix that represents projection onto W.

   c) Let $\vec{n} = N'$. Find the matrix A that describes the function

$$\text{refl}_W(\vec{v}) = \vec{v} - 2(<\vec{v}, \vec{n}> / <\vec{n}, \vec{n}>)\vec{n}.$$

   Hint : what happens on the standard unit vectors?

   d) What is $A\vec{x}$ for $\vec{x}$ in W?

   e) What is $A\vec{n}$?

   f) Give the eigenvalues and a basis of eigenvectors of A for $\mathbb{R}^3$.

SELECTED PROBLEMS FROM HOUR EXAMINATIONS FROM OTHER INSTRUCTORS

From Professor Wilkerson's practice test for the 2nd midterm:

1. Let W be the plane in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   1) Find and orthogonal basis for W.

   2) Find a vector in $\mathbb{R}^3$ which is perpendicular to W.

   3) Give an orthonormal basis of $\mathbb{R}^3$ so that each vector in your basis is either in W or $W^{\perp}$.

4) Give a (non zero) equation that each vector in W satisfies.

5) Find a 3×3 matrix P such that for each $\vec{v}$, $P\vec{v} = \text{Proj}_W(\vec{v})$, the projection of $\vec{v}$ onto W. Show that $P^2 = P$.

From Professor Yeoung's 2nd midterm:

2. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

(a). Find an *orthonormal* basis for the null space of A.

(b). Find an orthogonal basis for the row space of A.

(c). Find an orthogonal basis for $R(A')^{\perp}$, the orthogonal complement of the column space of A'.

(d). The union of the vectors from part (a) and part (b) should give you an orthogonal basis of $\mathbb{R}^3$. What are the coordinates of $[1,0,0]^t$ with respect to the basis? That is, solve c_1, c_2, c_3 in $[1,0,0]^t = c_1v_1 + c_2v_2 + c_3v_3$, where v_1, v_2, v_3 are the three vectors you obtained.

3. (a). Let Q be a 3×3 matrix satisfying $Q'Q = I_3$. Show that $QQ' = I_3$.

(b). If the columns of a 3×3 real valued matrix Q form an orthonormal basis of R^3, is it always true that the rows of Q form an orthonormal basis of R^3 as well? Why?

(c). Let

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Evaluate $P'P$.

(d). Find an **orthonormal basis** for the column space of P.

4. (a). Find the value of x such that 2 is an eigenvalue of

$$A = \begin{pmatrix} x & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b). Given that 1 is an eigenvalue of

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Find a basis for the eigenspace of A corresponding to eigenvalue 1.