The Pythagorean philosophy rested on the assumption that numbers were the substance of all things, i.e. they formed the basic organizing principle of the universe.

A constellation in the heavens could be characterized by the number of stars that compose it and by its geometrical form, which itself could be represented by a number. The motion of planets could be expressed in terms of ratios of numbers. Musical harmonies depended on numerical ratios.

Pythagoras and his followers took the first steps in the development of Number Theory and at the same time laid much of the basis of future number mysticism. For instance it is ascribed to Pythagoras the discovery of amicable, or friendly numbers.

Two numbers are amicable if each is the sum of the proper divisors of the other. 284 and 220 are amicable since:

* the proper divisors of 220 are
  1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110

and the sum of these is 284

* the proper divisors of 284 are 1, 2, 4, 71, 142 and the sum of these is 220.

In 1636, Pierre de Fermat discovered the next pair of amicable numbers: 17,296 and 18,416. René Descartes gave a third pair 2 years later. In 1747, Leonhard Euler undertook a systematic approach and gave a list of 30 pairs.
Even the famous violinist Niccoló Paganini found an overlooked and relatively small pair of amicable numbers: 1,184 and 1,210.

Other numbers having mystical connections to numerological speculations are the perfect numbers.

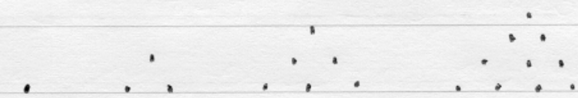
A number is perfect if it is the sum of its proper divisors. For instance

$$6 = 1 + 2 + 3$$
$$28 = 1 + 2 + 4 + 7 + 14$$
$$496 = \ldots\ldots$$

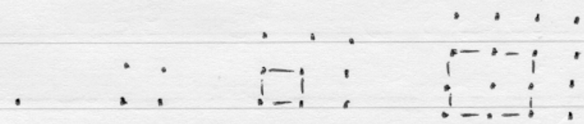
(e.g. God created the world in 6 days ---- a perfect number)

<u>Figurate numbers</u> also originated with the earliest members of the society. These numbers, considered as the number of dots in certain geometrical configurations, represent a link between geometry and arithmetic.

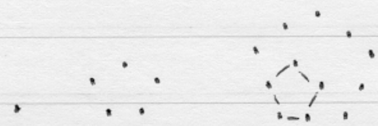
We have:

* triangular #s  etc

1 3 6 10

* square #s  etc

1 4 9 16

* pentagonal #s  etc...

1 5 12

Many interesting theorems can be established (in purely geometric fashion) concerning figurate numbers:

<u>Theorem</u>: any square number is the sum of two successive triangular numbers

e.g. 

3 6 9

6 10 16

Proof:
Nowadays we can establish the algebraic proof as follows:

the n-th triangular number $1+2+3+\dots+n = \overset{T_n}{\downarrow}\ \frac{n(n+1)}{2}$

Thus $T_{n-1} + T_n = \frac{(n-1)\cdot n}{2} + \frac{n(n+1)}{2} = n^2$.

$\blacksquare$

Theorem: the sum of any number of consecutive odd integers, starting with 1, is a perfect square.

Proof: algebraically again we have

$$1+3+5+\dots+(2n-1) = \frac{n(2n)}{2} = n^2$$

$\blacksquare$

Tradition is also unanimous in ascribing to Pythagoras the discovery of ... Pythagoras Theorem. This theorem was known to the Babylonians of Hammurabi's time, but the first general proof may well have been given by Pythagoras.

Closely allied to the Pythagorean Theorem is the problem of finding integers a, b, c that can represent the legs and hypothenuse of a right triangle:

$$a^2+b^2=c^2.$$

A tiple of numbers of this sort is known as a Pythagorean triple. We saw already that the analysis of Plimpton 322 offers evidence that the ancient Babylonians knew how to calculate them.

The Pythagoreans have been credited with the formulas:

$$* \quad m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2 \qquad \underline{m \text{ odd}}$$

$$* \quad n^2 + \left(\left(\frac{n}{2}\right)^2 - 1\right)^2 = \left(\left(\frac{n}{2}\right)^2 + 1\right)^2 \qquad n \ \underline{\text{even}}$$

Neither of these formulas yields all the Pythagorean triples.

Knowledge of the Pythagorean theorem led also to the discovery of what today are called <u>irrational numbers</u>. This discovery was surprising and disturbing to the Pythagoreans. First of all, it seemed to deal a mortal blow to the Pythagorean philosophy that all depends upon whole numbers. Next, it seemed contrary to common sense: it was felt intuitive that any magnitude could be expressed by some <u>rational</u> number. Geometically, no one would have doubted that for any two given line segments one is able to find some third line segment (perhaps very very small) that can be marked off a whole number of times into each of the 2 given segments.

However ... contrary to intuition there exist <u>incommensurable</u> (i.e. having no common unit of measure) <u>segments</u>.

<u>Theorem</u>: The side and diagonal of a square are incommensurable.

In our modern terminology the above statement is equivalent to

<u>Theorem</u>: $\sqrt{2}$ is irrational.

<u>Proof</u>: Suppose $\sqrt{2}$ is rational, i.e. $\sqrt{2} = p/q$ where we may assume that p and q have no common factors. Squaring both sides we get: $2 = p^2/q^2$ or $p^2 = 2q^2$. Thus p^2 is even. The only way this can happen is if p itself is even. Thus p^2 is actually divisible by 4. Hence q^2 and therefore q must be even. This is a contradiction to our assumption. $\blacksquare$