Square roots seem to have been found by formulas like these:

$$\sqrt{N} = \sqrt{a^2+h} \cong a + \frac{h}{2a} = \frac{1}{2}\left(a+\frac{N}{a}\right)$$

How did they discover it? There is some textual evidence that their formula begins from the identity

$$(x+y)^2 = x^2 + 2xy + y^2$$

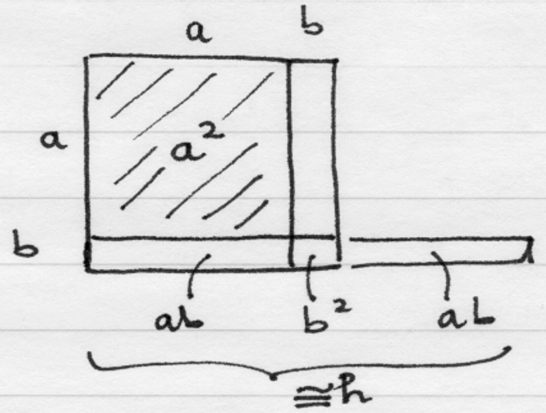
whose validity was discovered from its geometric equivalent. That is, given a square of area N they are seeking to compute the length of its side, $\sqrt{N}$.

To this end, they choose a regular number "a" ($\equiv$ regular means that it has a finite sexagesimal expansion) close to, but less than the desired result. After setting $h = N - a^2$, the next step is to choose b so that $2ab+b^2$ is as close as possible to h. In this way we have

$$\begin{cases} N = a^2 + h \approx a^2 + \underbrace{2ab+b^2}_{\approx h} = (a+b)^2 \\ \text{and} \quad \sqrt{N} \approx a+b \end{cases}$$

The geometric picture is:

square of area N



Now the scribes observes that if a^2 is close enough to N ... then $h = N - a^2$ is small. Thus in the relation $h = 2ab + b^2$ we have that b^2 will also be small in relation to $2ab$. Thus b can be chosen to the equal to $h/2a$:

$$h = 2ab + \underbrace{\cancel{b^2}}_{\text{small}} \approx 2ab \rightsquigarrow b = h/2a$$

and then $$\sqrt{N} = \sqrt{a^2+h} \approx a + b \cong a + h/2a$$

$$= a + \frac{N-a^2}{2a} = a + \frac{N}{2a} - \frac{a}{2}$$

$$= \frac{1}{2}a + \frac{N}{2a} = \frac{1}{2}\left(a + N/a\right).$$

Observe that the Babylonians got their approximation $\sqrt{2} \approx 1;25 = 1 + \frac{25}{60} \; (= 1\frac{5}{12})$
$= 1.41\overline{6}$

by choosing $a = 4/3$. Thus $a^2 = \frac{16}{9} = 1\frac{7}{9}$ so that $h = 2/9$ and hence

$$\sqrt{2} = \sqrt{\frac{16}{9} + 2/9} \approx 4/3 + \frac{2/9}{2 \cdot 4/3} = \frac{4}{3} + \frac{1}{12} = 1\frac{5}{12}$$

Another tablet was found where an even better approximation for $\sqrt{2}$ is given. In that tablet there is a square with size 30 and with other 2 numbers given

$1;24,51,10 \quad \left(\equiv 1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3} = 1.414212963\right)$

$42;25,35 \quad \left(\equiv 42+\frac{25}{60}+\frac{35}{60^2} = 42.4263889\right)$

It is reasonable to expect that the first was the approximation of $\sqrt{2}$ and the second one the length of the diagonal of the square $(=30\sqrt{2})$.

"<u>Calculus I comment:</u>" the approximation

$$\sqrt{a^2+h} \cong a+\frac{h}{2a}$$

is exactly the one we get nowadays in Calc I when we use linear approximation:

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x \; ;$$

in fact, if $f(x)=\sqrt{x}$ then $f'(x)=\frac{1}{2\sqrt{x}}$ and if we set $x_0=a^2$ and $\Delta x = h$ we get

$$f(a^2)=\sqrt{a^2}=a \qquad f'(a^2)=\frac{1}{2\sqrt{a^2}}=\frac{1}{2a}$$

$$\therefore \quad \sqrt{a^2+h} = f(a^2+h) \simeq a+\frac{1}{2a}\cdot h \, . \quad ■$$

All this discussion provides solid evidence of the interest of Babylonians in Pythagoras theorem. Even more substantial evidence is provided in the Babylonian tablet "Plimpton 322" (now at Columbia University).

This tablet was written in Larsa around 1800 BC.

In the remains of the tablet there are 4 columns of numbers, which — in our decimal system — are:

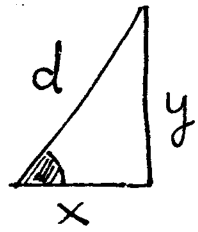
$\left(d/y\right)^2$	x	d	#	y
1.9834028	119	169	1	120
1.9491586	3367	4825 (11521)	2	3456
1.9188021	4601	6649	3	4800
1.8862479	12709	18541	4	13500
1.8150077	65	97	5	72
1.7851929	319	481	6	360
1.7199837	2291	3541	7	2700
1.6845877	799	1249	8	960
1.6426694	481 (541)	769	9	600
1.5861226	4961	8161	10	6480
1.5625	45	75	11	60
1.4894168	1679	2929	12	2400
1.4500174	161 (25921)	289	13	240
1.4302388	1771	3229	14	2700
1.3871605	56	106 (53)	15	90

Plimpton 322

The tablet contains in each row 2 of the numbers of a Pythagorean triple :

$x \equiv$ length of the short side

$d \equiv$ length of the hypothenuse



where as the number y [s/t $d^2 - x^2 = y^2$] was missing.
On the other hand the first column reports the quotients $(d/y)^2$.

There were 4 errors in the tablet. They correspond to the numbers put in parenthesis. Three of the errors can be easily accounted for. In line 10 we should have had 481 instead of 541. Notice that $481 = 8 \cdot 60 + 1$ whereas $541 = 9 \cdot 60 + 1$. That is, there was a mere slip of the stylus when writing these numbers in cuneiform form.
Notice that $25921 = 161^2$, whereas $53 = 106/2$.

All the triples in the Plimpton 322 tablet have another remarkable feature. Except the ones on line 11 and 15, all the triples are primitive Pythagorean triples. I.e. $\gcd(x, y, d) = 1$.

For instance (3,4,5) is a primitive triple, whereas (6,8,10) is <u>not</u> since the gcd of those numbers is 2.

We recall that all <u>primitive Pythagorean triples</u> are of the form:

$$x = u^2 - v^2 \qquad y = 2uv \qquad d = u^2 + v^2$$

where u, v are relative prime numbers and $u > v$.

The first column contains the values of $\left(\frac{d}{y}\right)^2$ for different triangles. This column is a table giving the square of the secant of the angle opposite to the side y of the previous triangle.

It actually seem to be the table of the square of the secant for angles from 45° to 31°.

Overall the fact remains that scribes were well aware of the Pythagorean Theorem.

<u>Solving Equations</u>: The Babylonians solved linear and quadratic equations in 2 variables, and even problems involving cubic and biquadratic equations.