Cardano and the Solution of the Cubic

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In 1494, the Italian Luca Pacioli produced his volume titled *Summa de Arithmetica*. In this a step was made in the direction of symbolic algebra in which he treated the standard mathematics of his day with emphasis on solving both linear and quadratic equations. He decided that the cubic was quite impossible to solve, and thus laid out a challenge to the Italian mathematical community to find a solution.
Scipione del Ferro

del Ferro, of the University of Bologna, decided to take up the challenge.

He discovered a formula that solved the so-called "depressed cubic" of the form:

\[ ax^3 + cx + d = 0 \]

Instead of publishing his solution, del Ferro kept it a secret until his deathbed, telling his student Antonio Fior.
Niccolò Fontana - Tartaglia

- Fior, with his new weapon, leveled a challenge against the Brecian scholar Niccolò Fontana, also known as Tartaglia in 1535.
- Tartaglia claimed to know the solution to cubics of the form: $ax^3 + bx^2 + d = 0$.
- Tartaglia sent Fior a list of 30 various mathematical problems and Fior in turn sent Tartaglia a list of 30 depressed cubics, placing Tartaglia in a bind.
- He worked furiously trying to find a solution to these depressed cubics and on the night of February 13, 1535, he discovered the solution.
- Tartaglia prevailed in the challenge.
Tartaglia’s Poem

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different
By that one; this feat
Will be kept as a rule
Their product always equal, the same,
To the cube of a third
Of the number of things named.
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count.
when a cube and its things near
Add to a new number, discrete

This means to get rid of the $x^2$ term

Substitute $x = \frac{y - a}{3}$

$a = \text{the coefficient of the } x^2 \text{ term}$

Insert this $x$ into the original equation
Determine two new numbers different By that one

- Substitute \( y = \)
- \( a = \)
- \( p = \) the coefficient of the \( y \) term
- Insert \( y \) into previous equation
- Transform equation into a quadratic equation by multiplying by \( w^3 \)
- Solve for \( w^3 \)
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count

- Take the cubed root of what you found \( w^3 \) to equal
- These are your \( a \) and \( \beta \) values
- With these values, you can solve for your three equations
  - \( a + \beta \)
  - \( a + b\beta \)
  - \( b + a\beta \)
- Where \( a = \frac{-1 + \sqrt{3}}{2} \)
- And \( b = \frac{-1 - \sqrt{3}}{2} \)
- These values are your \( y \)-values for your first equation, which will solve for your 3 roots
Most Bizarre Character

Gerolamo Cardano of Milan then entered into the scene, considered by some to be the most bizarre character in the whole history of mathematics.
Gerolamo Cardano was born in Pavia in 1501 as the illegitimate child of a jurist. He attended the University of Padua and became a physician in the town of Sacco, after being rejected by his home town of Milan. He became one of the most famous doctors in all of Europe, having treated the Pope. He was also an astrologer and an avid gambler, to which he wrote the *Book on Games of Chance*, which was the first serious treatise on the mathematics of probability.
But his passion was studying, teaching, and writing mathematics.

Hearing of Tartaglia’s discovery of the depressed cubic, Cardano wrote to him numerous times begging to be told of the solution.

Tartaglia finally revealed the secret to Cardano on March 25, 1539, with the oath of Cardano to never publish the findings.
Cardano, along with his servant/pupil/colleague Ludovico Ferrari, discovered the solution of the general cubic equation:

\[ x^3 + bx^2 + cx + d = 0 \]

But his solution depended largely on Tartaglia’s solution of the depressed cubic and was unable to publish it because of his pledge to Tartaglia.

In addition, Ferrari was also able to discover the solution to the quartic equation, but it also required the use of the depressed cubic.
In 1543, Cardano and Ferrari traveled to Bologna.

While inspecting del Ferro’s papers, they came across a solution to the depressed cubic in del Ferro’s own handwriting.

Cardano and Ferrari were no longer prohibited from publishing their results.
In 1545, Cardano published his book *Ars Magna*, the “Great Art.”

In it he published the solution to the depressed cubic, with a preface crediting del Ferro with the original solution.

He also published his solution of the general cubic and also Ferrari’s solution of the quartic.
The Solution to the Cubic
Cardano’s Method

\[ x^3 + ax^2 + bx + c = 0 \]

Eliminate the square term by using the substitution \( x = t - \frac{a}{3} \)

\[ t^3 + pt + q = 0 \]

where \( p = b - \left(\frac{a^2}{3}\right) \) and \( q = c + \left(\frac{2a^3 - 9ab}{27}\right) \)
Letting \( t = u+v \), rewrite the above equation as

\[ u^3 + v^3 + (u+v)(3uv + p) + q = 0 \]

Next we set \( 3uv + p = 0 \), and the above equation becomes \( u^3 + v^3 = -q \)

And we are left with these two equations:

\[ u^3 + v^3 = -q \]
\[ u^3v^3 = -p^3/27 \]
Since the below equations are the product and sum of $u^3$ and $v^3$ then there is a quadratic equation with roots $u^3$ and $v^3$

\[ u^3 + v^3 = -q \]
\[ u^3v^3 = -p^3/27 \]

This quadratic equation is
\[ t^2 + qt - p^3/27 \]
with solutions
\[ u^3 = (-q/2) + \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}} \]
\[ v^3 = (-q/2) - \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}} \]
\[ u^3 = (-q/2) + v(q^2/4) + (p^3/27) \]
\[ v^3 = (-q/2) - v(q^2/4) + (p^3/27) \]

Next we must find the cube roots of these equations to solve for \( u \) and \( v \)

If \( 27q^2 + 4p^3 < 0 \)
the roots will be complex numbers

If \( 27q^2 + 4p^3 > 0 \)
Then the roots will not be complex