Cardano and the Solution of the Cubic

Bryan Dorsey, Kerry-Lyn Downie, and Marcus Huber

Pacioli

- In 1494, the Italian Luca Pacioli produced his volume titled *Summa de Arithmetica.*
- In this a step was made in the direction of symbolic algebra in which he treated the standard mathematics of his day with emphasis on solving both linear and quadratic equations.
- He decided that the cubic was quite impossible to solve, and thus laid out a challenge to the Italian mathematical community to find a solution.

Scipione del Ferro

- del Ferro, of the University of Bologna, decided to take up the challenge.
- He discovered a formula that solved the so called “depressed cubic” of the form:
- $ax^3 + cx + d = 0$
- Instead of publishing his solution, del Ferro kept it a secret until his deathbed, telling his student Antonio Fior.

Niccolo Fontana - Tartaglia

- Fior, with his new weapon, leveled a challenge against the Brecian scholar Niccolo Fontana, also know as Tartaglia in 1535.
- Tartaglia claimed to know the solution to cubics of the form: $ax^3 + bx^2 + d = 0$.
- Tartaglia sent Fior a list of 30 various mathematical problems and Fior in turn sent Tartaglia a list of 30 depressed cubics, placing Tartaglia in a bind.
- He worked furiously trying to find a solution to these depressed cubics and on the night of February 13, 1535, he discovered the solution.
- Tartaglia prevailed in the challenge.

Tartaglia’s Poem

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different
By that one; this feat
Will be kept as a rule
Their product always equal, the same,
To the cube of a third
Of the number of things named.
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count.

when a cube and its things near
Add to a new number, discrete

- This means to get rid of the x^2 term
- Substitute x = $y - \frac{a}{3}$
- a = the coefficient of the x^2 term
- Insert this x into the original equation

Determine two new numbers different By that one

- Substitute y = $w + \frac{a}{w}$
- a = $\frac{-p}{3}$
- p = the coefficient of the y term
- Insert y into previous equation
- Transform equation into a quadratic equation by multiplying by w^3
- Solve for w^3

Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count

- Take the cubed root of what you found w^3 to equal
- These are your a and ß values
- With these values, you can solve for your three equations
- a + ß
- *a* a + bß
- *b* a + aß
- Where a = $\frac{-1+\sqrt{-3}}{2}$
- And b = $\frac{-1-\sqrt{-3}}{2}$
- These values are your y-values for your first equation, which will solve for your 3 roots

Most Bizarre Character

- Gerolamo Cardano of Milan then entered into the scene, considered by some to be the most bizarre character in the whole history of mathematics.

Cardano

- Gerolamo Cardano was born in Pavia in 1501 as the illegitimate child of a jurist.
- He attended the University of Padua and became a physician in the town of Sacco, after being rejected by his home town of Milan.
- He became one of the most famous doctors in all of Europe, having treated the Pope.
- He was also an astrologer and an avid gambler, to which he wrote the *Book on Games of Chance*, which was the first serious treatise on the mathematics of probability.

- But his passion was studying, teaching, and writing mathematics.
- Hearing of Tartaglia's discovery of the depressed cubic, Cardano wrote to him numerous times begging to be told of the solution.
- Tartaglia finally revealed the secret to Cardano on March 25, 1539, with the oath of Cardano to never publish the findings.

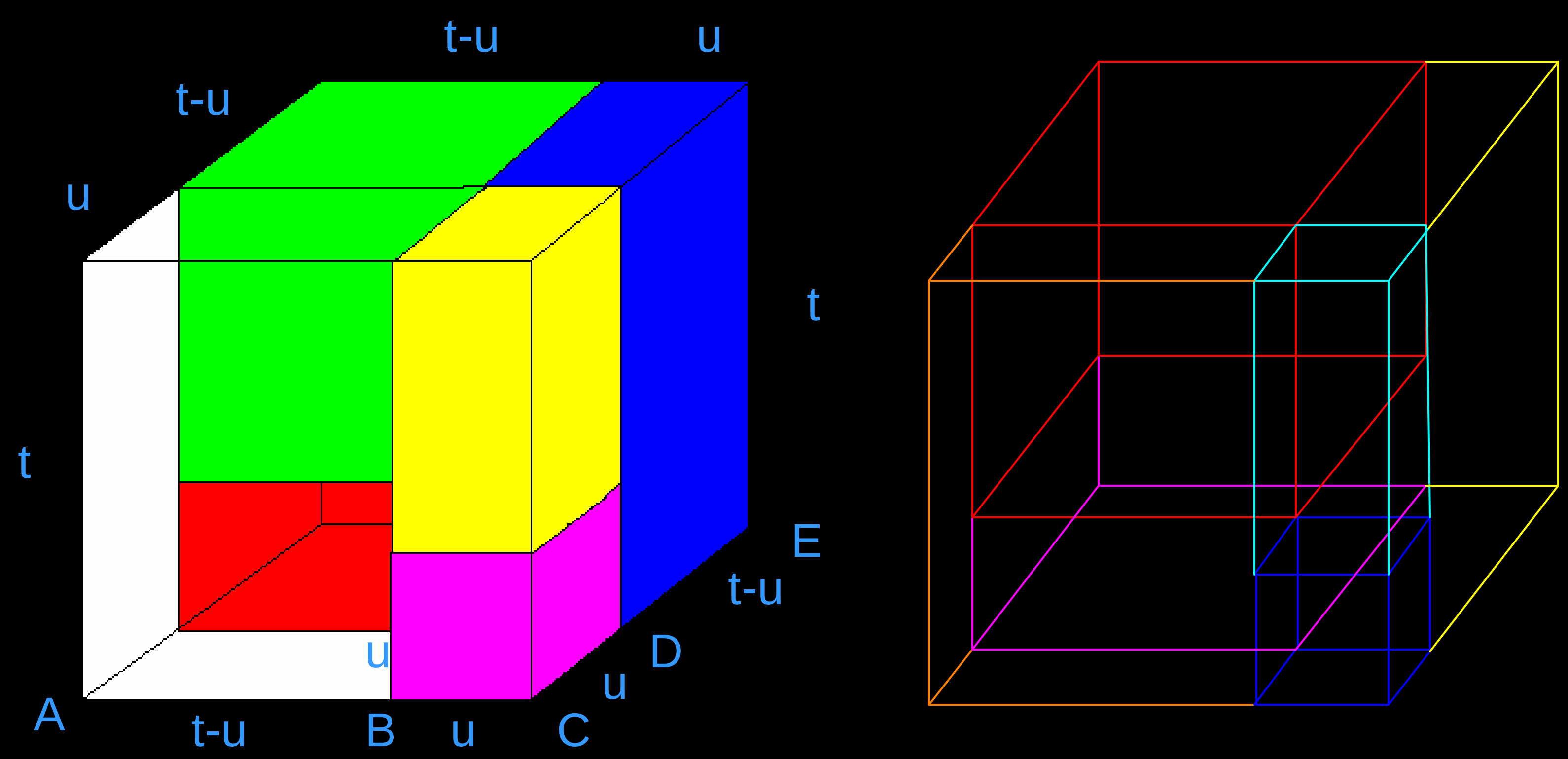
- Cardano, along with his servant/pupil/colleague Ludovico Ferrari, discovered the solution of the general cubic equation:
- $x^3 + bx^2 + cx + d = 0$
- But his solution depended largely on Tartaglia’s solution of the depressed cubic and was unable to publish it because of his pledge to Tartaglia.
- In addition, Ferrari was also able to discover the solution to the quartic equation, but it also required the use of the depressed cubic.

- In 1543, Cardano and Ferrari traveled to Bologna.
- While inspecting del Ferro’s papers, they came across a solution to the depressed cubic in del Ferro’s own handwriting.
- Cardano and Ferrari were no longer prohibited from publishing their results.

Ars Magna

- In 1545, Cardano published his book *Ars Magna*, the “Great Art.”
- In it he published the solution to the depressed cubic, with a preface crediting del Ferro with the original solution.
- He also published his solution of the general cubic and also Ferrari’s solution of the quartic.

The Solution to the Cubic



Cardano's Method

$$x^3 + ax^2 + bx + c = 0$$

Eliminate the square term by using the substitution $x = t - a/3$

$$t^3 + pt + q = 0$$

where $p = b - (a^2/3)$ and $q = c + (2a^3 - 9ab)/27$

Letting t = u+v, rewrite the above equation as

$$u^3 + v^3 + (u+v)(3uv + p) + q = 0$$

Next we set $3uv + p = 0$, and the above equation becomes $u^3 + v^3 = -q$

And we are left with these two equations:

$$u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27$$

Since the below equations are the product and sum of u^3 and v^3 then there is a quadratic equation with roots u^3 and v^3

$$u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27$$

This quadratic equation is

$$t^2 + qt – p^3/27$$

with solutions

$$u^3 = (-q/2) + v(q^2/4) + (p^3/27)$$

$$v^3 = (-q/2) - v(q^2/4) + (p^3/27)$$

$$u^3 = (-q/2) + v(q^2/4) + (p^3/27)$$

$$v^3 = (-q/2) - v(q^2/4) + (p^3/27)$$

Next we must find the cube roots of these equations to solve for u and v

If $27q^2 + 4p^3 < 0$

the roots will be complex numbers

If $27q^2 + 4p^3 > 0$

Then the roots will not be complex