Euler's Refutation of Fermat's Conjecture

Introduction of Fermat

- Fermat, a French mathematician of the 17th century, is known as the founder of modern number theory.
- He discovered the principle of analytic geometry. He is also known as the inventor of differential calculus. Along with his friend Pascal, he invented the theory of probability.
- Fermat was a famous 17th century mathematician who founded modern number theory.

Biography

- Pierre de Fermat was born on August 17, 1601 in Beaumont-de Lomagne, France.
- His family consisted of his father, who was a wealthy leather merchant, a brother, and two sisters.
- After his early education at a local school, he attended the University of Toulouse, where he studied law until the second half of the 1620s.

Biography

- At this time he moved to Bordeaux where, following the custom of his day, Fermat began a reconstruction of Apollonius's *Plane Loci* in 1629. Also during this time he did work on minima and maxima which he gave to Etienne d'Espagnet.

- From here Fermat went to Orleans, where he studied law at the University, and received his degree in civil law.

Biography

- Back in Toulouse, he first was appointed to the lower chamber of the parliament in 1631, and rose through the ranks over until he was appointed to a position in the Criminal Courts in 1638. He lived here for the remainder of his life.

- In the early 1650's the plague struck the region killing many of the older men in the Criminal Court, Fermat received more promotions.

- He also, was one of the many struck by it.

- He survived, though his death was wrongly reported in 1653. However, it was soon corrected.

Biography

- During this time, Fermat was preoccupied with his hobby, mathematics.
- He had a friendship with Beugrand (a French mathematician) and Carvie (a French amateur mathematician). Through Carvie, Fermat met Mersenne, who was interested in Fermat's discoveries on falling bodies .
- Mersenne, (a French monk who is best known for his role as a cleaning house for correspondence between eminent philosophers and scientists and for his work in number theory).

Biography

- Fermat's correspondence with his colleagues faded it 1643, but came back in 1654.

Reasons:

- Pressure of work kept him from devoting so much time to mathematics.

- The Fronde, a civil war in France, took place and from 1648 Toulouse was greatly affected.

Biography

- Not until Blaise Pascal, wrote him concerning his ideas on probability. They had a short correspondence about the theory, and that is why they are both credited for it.

- Correspondences with other mathematicians continued until Fermat died on January 12, 1665.

Biography

- Although Fermat made many contributions to the world of mathematics, his influence was severely hampered by his refusal to have anything published.

Contributions to Mathematics

- Throughout the course of his life, Fermat came up with many mathematical theorems and played significant roles in geometry and calculus.

- He began his math studies by restoring lost works of antiquity.

- His work on the *Plane Loci*, or Greek geometry of the 3rd century B.C. found that the study of Loci, or sets of points with certain characteristics, could be done by the application of algebra to geometry through the coordinate system .

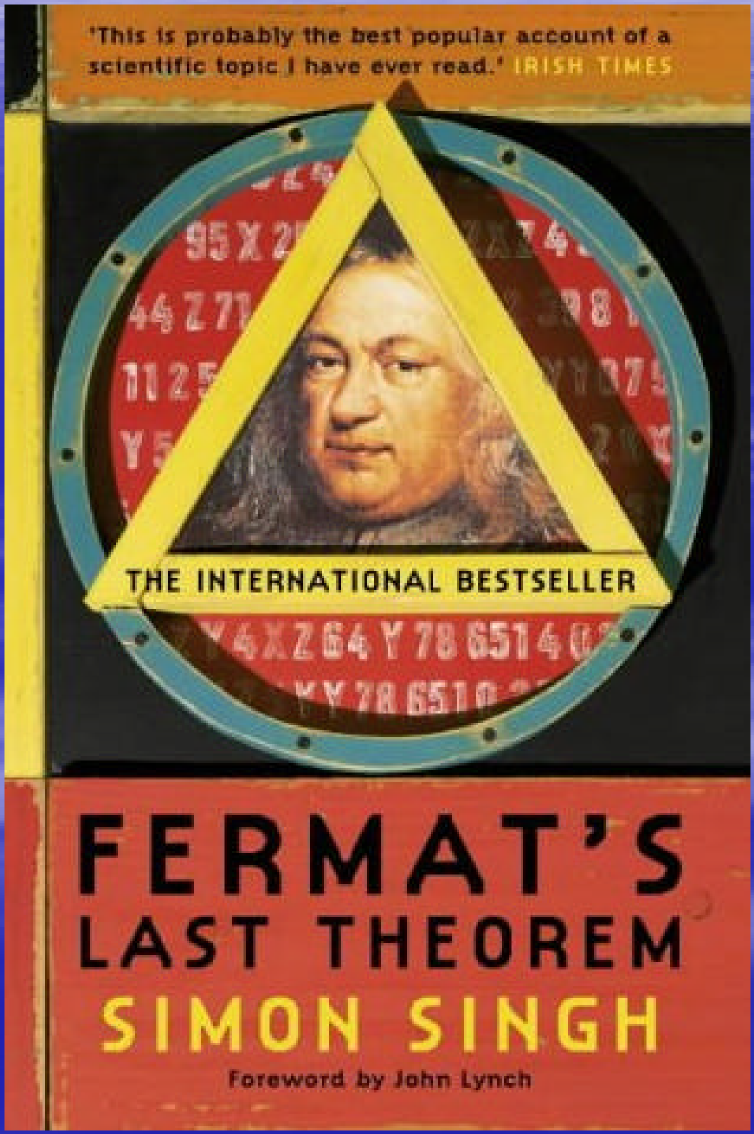
Contributions to Mathematics

- This work was published in 1679, and became known as Cartiesian geometry.

- Fermat's favorite field of study was numbers, “Particularly Number Theory”.

Last Theorem

- Of all of his work, Fermat is best known for his "Last Theorem" .
- This is quite ironic because his own proof for it has never been found.
- Some say that Fermat never really had a proof, or that his proof was wrong.



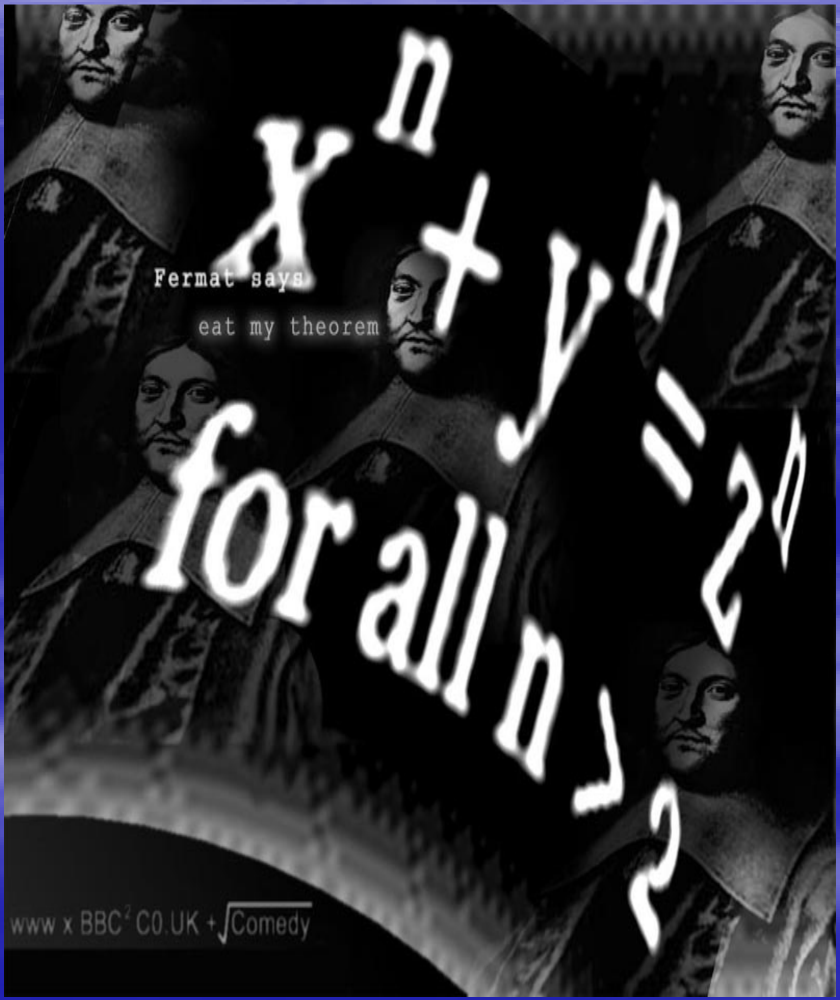
Last Theorem

- The theorem to which he was referring is:

  $x^n + y^n = z^n$, has no integer solutions for *x*, *y*, and *z* when $n > 2$.

- Fermat's Last Theorem has some of its roots in the Pythagorean Theorem.

- Fermat's Last Theorem" has eluded mathematicians for over 300 years.



Last Theorem

- A researcher at Princeton, Andrew Wiles, claimed to have a proof for Fermat's Last Theorem during a three day seminar.

- However, the proof was found to be wanting, and on October 25, 1994, the professor and his coworker Richard Taylor released two preprints *Ring theoretic properties of certain Hecke algebras* and *Modular elliptic curves and Fermat's Last Theorem* .

Last Theorem

- **Since then other current mathematicians, have simplified Wiles' argument and it is now generally accepted.**
- However Fermat's "original proof" still has not been found and probably never will.
- If nothing else Fermat's Last Theorem has succeeded in doing one thing, annoying a bunch of mathematicians for 300 years.

Last Theorem

- Perhaps this is what Fermat intended all along.

Life of Euler

- Born on the 15th April 1707, Leonhard Euler was the son of a Lutheran Minister.

- He was given a simple, rather poor, education in his home town of Basel, Switzerland, before entering the University of Basel at the age of 14.

Life of Euler

- He always had a deep interest in mathematics, reading books on the subject from an early age.

- Whilst growing up he had come into contact with Johann Bernoulli, a family friend and impressive mathematician.

Life of Euler

- Entered the 1727 grand prize of the Paris Academy. Despite only coming second it was a remarkable achievement for one so young.
- He took up the post in the mathematical-physical division of the Academy in May 1727.
- He served for 3 years before being made professor of physics in 1730.

Life of Euler

- Euler worked for many years on projects at the academy.
- Mainly studied number theory, differential equations and rational mechanics.
- Won the grand prize of the Paris Academy in 1738 and 1740 he was now a very well respected mathematician.

Life of Euler

- He was suffering health issues, nearly died of fever in 1735 and lost the sight in one eye.

- In 1766 Euler left Berlin to return to St. Petersburg, shortly after his return to Russia he became almost totally blind .

Life of Euler

- He was still able to continue working right up to his death. Indeed, he published many articles after he became blind with the help of friends and family.
- Euler died on 18th September 1783, possibly of a brain hemorrhage.

Fermat to Euler via Goldbach

- As we have seen, Fermat did extraordinary work in the field of Number Theory, but much of his work was in the from of unproven assertion.
- It was Christian Goldbach (of he famous Goldbach Conjecture), a fervent proponent of Number Theory, who brought these assertions to Euler’s attention.

Fermat to Euler via Goldbach

- At first, Euler was reticent to pursue the subject of Number Theory. However, his own curiosity and Goldbach’s constant prodding led Euler to investigate.
- Before he finished, Euler’s number theory would fill four large volumes of his work Opera Omnia.

Fermat to Euler via Goldbach

- It has been noted that these four volumes alone would firmly place Euler in the pantheon of great mathematicians.

Example of Euler's Work on Fermat

- He investigated primes that can be written as the sum of two perfect squares.
- Besides 2, all primes are odd numbers.
- When an odd number is divided by 4 one obtains a remainder of either 1 or 3.

Example of Euler's Work on Fermat

- I.e. if $p>2$ and p is prime:

$$p = 4k + 1$$

or

$$p = 4k + 3$$

for some whole number k.

Example of Euler’s Work on Fermat

- Fermat asserted that primes of the form:

  $$p = 4k + 1$$

  can be written as the sum of two perfect squares in one and only one way.

- And, primes of the form:

  $$p = 4k + 3$$

  can not be written as the sum of two perfect squares.

Example of Euler's Work on Fermat

- I.e.:

$$193 = (4 \times 48) + 1$$

This can be written as:

$$193 = 144 + 49 = 12^2 + 7^2$$

which is unique.

Example of Euler's Work on Fermat

And,

$$199 = (4x49) + 1$$

can not be written as the sum of two squares.

A Little More of Euler and Fermat

- Another of Fermat's assertion which gained Euler's interest appeared in 1640.

  Fermat stated if a is any whole number and p is a prime not a factor of a. Then, p must be a factor of:

$$a^{p-1} - 1$$

A Little More of Euler and Fermat

- This result became known as:

Fermat’s Little Theorem

A Little More of Euler and Fermat

- This theorem as we shall see was proved by Euler in 1736.

Proof of Little Fermat Theorem

- I. If a prime divides a product then it divides one of the factors. This was proved by Euclid in his book (Elements).
- II. If p is prime and a is any whole number, then

$$a^{p-1} + \frac{p-1}{2\times 1}a^{p-2} + \frac{(p-1)(p-2)}{3\times 2\times 1}a^{p-3} + \ldots\ldots + a$$

Proof of Little Fermat Theorem

- Finally, the binomial theorem for $(a+1)^p$.
- Theorem 1: if p is prime and a is any whole number, then $(a+1)^p - (a^p+1)$ is evenly divisible by p.

$$(a+1)^p = [a^p + pa^{p-1} + \frac{p(p-1)}{2\times 1}a^{p-2}$$

$$+\frac{p(p-1)(p-2)}{3\times 2\times 1}a^{p-3} + ...pa + 1]$$

Proof of Little Fermat Theorem

- Theorem 2: If p is prime and if $a^p - a$ is evenly divisible by p, then so is

  $$(a+1)^p - (a+1) .$$

- Proof: Theorem 1 tells us that p divides evenly into $(a+1)^p - (a^p+1)$. By assumption, p also divides evenly into $a^p - a$. Thus p divides into sum of these two

$$[(a+1)^p - (a^p + 1)] + [a^p - a] = (a+1)^p - a^p - 1 + a^p - a$$

$$= (a+1)^p - (a+1)$$

Proof of Little Fermat Theorem

- Euler now proved the Little Fermat Theorem by mathematical induction.
- Theorem 3: If p is prime and a is any whole number, then p divides evenly into $a^p - a$.
- Proof: $a = 1$: $a^p - a = 1^p - 1 = 1 – 1 = 0$, P divided evenly into 0 since all whole numbers do.
- By applying Theorem 2 for $a = 1$,

$$(1+1)^p - (1+1) = 2^p - 2.$$

Euler showed that p is also a divisor for $a = 2$.

Proof of Little Fermat Theorem

- Next Euler used Theorem 2 for a = 2:

  $$(2+1)^p - (2+1) = 3^p - 3.$$

  Hence,

  The result holds for a = 3.

- Repeating this process, Euler found that this holds for any whole number a. So, p is a factor of $a^p - a$.

Proof of Little Fermat Theorem

- Little Fermat Theorem: If p is prime and a is a whole number which does not have p as a factor, then p divides evenly into $a^{p-1} - 1$.

- Proof: we have just shown

$$a^p - a = a(a^{p-1} - 1)$$

Proof of Little Fermat Theorem

- Since p is a prime, I implies that p must divide evenly into either a or $a^{p-1} - 1$ (or both).

  But

  by assumption, p does not divide evenly into a. Thus, p divides into $a^{p-1} – 1$.

Euler's Refutation of Fermat's Conjecture

- Conjecture: $2^{2^n} + 1$ is a prime.
- Theorem A: Suppose a is an even number and p s a prime that is not a factor of a but that does divide evenly into a + 1. Then for some whole number k,

  p = 2k + 1.

  Proof:

  a is even, so a + 1 is odd, but by assumption p divides evenly into a + 1, so p is odd. So, p -1 is even: p – 1 = 2k, thus p = 2k + 1.

Euler's Refutation of Fermat's Conjecture

- Theorem B: Suppose a is an even number and p is a prime that is not a factor of a but such that p does divide evenly into $a^2 + 1$. Then for some whole number k, $p = 4k + 1$.

- Proof: Shown on B.B.

Euler's Refutation of Fermat's Conjecture

- Theorem C: Suppose a is an even number and p is a prime that is not a factor of a but such the p does divide evenly into $a^4 + 1$. Then for some whole number k, $p = 8k + 1$.

- Proof:

Shown on B.B.

Euler's Refutation of Fermat's Conjecture

- Theorem: $2^{32} + 1$ is not prime.

Proof: Shown on B.B.

$2^{32} + 1 = 4{,}294{,}967{,}297 = 641 \times 6{,}700{,}417$

Fermat was wrong!!!!!!!

THE END!!!!!!

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