The Extraordinary Sums of Leonhard Euler

Your Presenters

- History/Biography: Aaron Boggs
- Infinite Series: Jonathan Ross
- Number Theory: Paul Jones

Overview

- First, and most importantly, Euler is pronounced “oiler”.
- Euler was beyond any doubt an absolute genius.
- Euler was one of the most, if not the the most, prolific mathematicians of all time ranging from basic algebra and geometry to complex number theory.

Euler's Place in this world

- ♥Euler was born in Switzerland in the city of Basel in 1707. (The same city as the famous Bernoullis)
- ♥Euler's father was a Calvinist preacher who had had some instruction by the famous Jakob Bernoulli.
- ♥Using his influence and connection to the Bernoullis his father arranged for Euler to study under Johann Bernoulli.

Euler's Early life

- ♠Johann Bernoulli was a tough teacher and was easily irritated by his pupils, including Euler.
- ♠But even this early in Euler's life, Johann could see that Euler had a talent for mathematics.
- ♠While still in his teens Euler was publishing high quality mathematical papers.
- ♠And at age 19, Euler won a prize from the French Academy for his analysis of the optimum placement of masts on a ship.
- ♠Note: at this point in time, Euler had not seen a ship.

Moving on Up

- About this time, in 1725, Johann Bernoulli's son Nikolaus II moved to St. Petersburg where his other son Daniel was working for the St. Petersburg Academy.
- There were no math spots open at the time of Euler's arrival, so he had to begin with an appointment in medicine and physiology in 1727.
- After some bouncing around from different positions, Euler got a mathematical chair at the Academy in 1733 when Daniel Bernoulli returned to Switzerland.

Where's Euler?

- 1725-1741: St. Petersburg Academy. During this time there were four Czars of Russia. Tired of the Czarist system, Euler took a position in Berlin.
- 1741-1766: Berlin Academy. Here Euler served under Frederick the Great. Their personalities didn't go well together. Frederick called Euler a "mathematical cyclops". He left.
- 1766-1789: St. Petersburg Academy. Catherine the Great served as Empress over these last years of Euler's life.

Euler’s Amazing Life

Hardship

- In 1735, while in St. Petersburg, Euler lost sight in his right eye. This is why Frederick called him a “mathematical cyclops”. In 1760, Euler lost sight in his left eye. From 1760-1789 Euler was completely blind.
- This did not stop Euler, in fact, it hardly hindered him. Euler simply did his math in his head and continued to dictate many papers and books while blind.
- “Just as deafness proved no obstacle to Ludwig von Beethoven a generation later, so blindness did not reduce the flow of mathematics from Leonhard Euler.”

Euler’s Amazing Life

Memory

- ♥Euler’s amazing memory greatly aided in his ability to work blind.
- ♥Some items Euler memorized:
  - ✓The first 100 prime numbers
  - ✓All their squares, cubes, and fourth, fifth, and sixth powers
  - ✓Could hold at least 50 places of accuracy while doing mental calculations
  - ✓The entire text of Virgil’s *Aeneid*, 63 719 words long, in latin, for at least 50 years

Euler's Amazing Life

Most prolific mathematician

- ♥Euler was one of, if not the, most prolific author of mathematics of all time.
- ♥Euler's published works include 560 books, papers, and memoirs during his life. After his death, from his left manuscripts, there was an estimated 47 year publication backlog. This increased the number to 856, and there were also 31 works written by Euler's eldest son, Johann, with Euler's supervision.
- ♥These 887 works account for approximately 1/3 of the total worldwide mathematical science works during Euler's lifetime.

Euler’s Amazing Life

Mathematical applications

♥Aiding in Euler’s vast quantity of works was the amazing variety of subjects he wrote about.

♥In addition to writing textbooks describing elementary mathematics through calculus, Euler also wrote about:

- ♥Optics: telescopes, microscopes, etc.
- ♥Engineering: ships, artillery, hydraulics
- ♥Acoustics: music
- ♥Astronomy: planetary motion
- ♥Physics: Newton’s dynamics

Euler's Notations

- Not only did Euler write more than anyone else, he also did it in such a way that it could be easily followed.
- Euler's notation in *Introductio in Analysin* (1748) was adopted by such notable mathematicians as Lagrange, Laplace, and Gauss and is still used today.
- Gauss said of Euler, "The study of Euler's works will remain the best school for the different fields of mathematics and nothing else can replace it"
- This statement has proved true as we continue to use his notations, solving, and teaching methods today.
- We owe to Euler the notation $f(x)$ for a function (1734), e for the base of natural logs (1727), i for the square root of -1 (1777), p for pi, Σ for summation (1755), the notation for finite differences, sine, cosine, tangent, and many others.

Euler's Family Life

- Mathematics and science were not the only fields in which Euler was prolific.
- Euler had 13 children throughout his life by two wives
- Euler was known as a kind and generous man who loved his family and he maintained a little garden.
- On September 18, 1783 while in St. Petersburg, Euler calculated laws for the ascent of balloons and also outlined the calculation of the orbit of the recently discovered planet Uranus. Later that day he was playing with his grandson, and while they played he had a stroke.
- Sources report his last words to be "I die".

Onto Euler's Sums

One of the first acheivements was the solution to the series:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \ldots$$

This series had stumped all the world's leading mathematicians including Jakob and Johann Bernoulli and Leibniz.

The solution that follows solidified Euler's reputation as a mathematical powerhouse.

Euler's Wonderful Sums:

The Infinite P-Series

When the 17th century began, little was known about infinite series.

This is possibly due to the fact that they were not frequently encountered.

- ➢But when infinite series were found, many leading mathematicians of the time could not find a solution.

- ➢"The Basel Problem" was one such stumper of the day baffling the greatest of mathematicians including Jakob Bernoulli.

➢The Basel Problem➣

The Basel problem amounts to finding the exact sum to the infinite series of:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\frac{1}{k_n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$$

Euler began his study of the Basel problem by direct numerical approximation, to see where it looked like it might be going.

But the series converges so slowly, that these approximations provided little insight.

$$= 1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{100} = 1.54977 \quad \textit{ten} \quad \textit{terms}$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{1000} = 1.63498 \quad \textit{one} \quad \textit{hundred} \quad \textit{terms}$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \ldots \frac{1}{1000^{2}} = 1.64393 \quad \textit{one} \quad \textit{thousand} \quad \textit{terms}$$

In fact, this one thousand term result is only correct to two decimal places!

Euler's Solution to Basel's Problem

Step 1:

First Euler found that the series could be redefined as:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 \cdot 2^{k-1}} + \left[\ln(2)\right]^2$$

This series could be used to find a much better approximation as it is a more rapidly converging series.

To only 14 terms Euler was able to approximate the series as:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 \cdot 2^{k-1}} + \left[\ln(2)\right]^2 = 1.644934$$

This solution is accurate to six decimal places.

End of Day One

Euler's Solution to Basel's Problem

Step 2:

Next, Euler went through a series of computations, rearranging of formulas (the details will be give in class at the blackboard!), as well as made a few risky assumptions to arrive at the amazing final answer of:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{p^2}{6}$$

Euler couldn’t solve just one infinite series…

After his success with the Basel Problem, Euler turned his attention to find the exact sum of p-series with p > 2. That is infinite series of the form:

$$\sum_{k=1}^{\infty} \frac{1}{k^{p}} \quad , \quad p > 2$$

He was able to do this for values of *p* that were even by proving his theorem that showed…

$$\sum_{k=1}^{n} r_k = A$$

$$\sum_{k=1}^{n} r_k^2 = A\sum_{k=1}^{n} r_k - 2B$$

$$\sum_{k=1}^{n} r_k^3 = A\sum_{k=1}^{n} r_k^2 - B\sum_{k=1}^{n} r_k + 3C$$

$$\sum_{k=1}^{n} r_k^4 = A\sum_{k=1}^{n} r_k^3 - B\sum_{k=1}^{n} r_k^2 + C\sum_{k=1}^{n} r_k - 4D$$

Euler & Number Theory

Recall

- Important to recall Proposition 20 of Book IX of Euclid's Elements. A finite collection of primes can never include all primes.

- Let p_1, p_2, …., p_n be any finite set of primes. Let $M = (p_1 \times p_2 \times ... p_n) + 1$ and consider the following options:

  Option A: If M is prime, then it surely is a "new" prime not in the original set, since it is certainly larger then p_1, p_2, …., p_n.

  Option B: If M is composite it must have a prime factor q, q is not one of the original primes. For if $q=p_k$ for some k, then q would divide evenly into both M and $p_1 \times p_2 \times ... p_n$ and then into the their difference $M - p_1 \times p_2 \times .... \times p_n = 1$. But the prime q, being at least 2, cannot divide evenly into 1. This contradiction means that q, differing from all p is the new prime.

Number Theory and Infinite Series

One Less than Perfect-

- In Euler's paper **"Variae observationes circa series infinitas,"** he investigates the summing of the infinite series. He proposed to find the exact sum of the series:

$$\frac{1}{15}+\frac{1}{63}+\frac{1}{80}+\frac{1}{255}+\ldots$$

- Uses his result seen earlier, with a lot of manipulation, Euler equates:

$$\frac{1}{15}+\frac{1}{63}+\frac{1}{80}+\frac{1}{255}=\left[\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\ldots\right]-\left[\frac{1}{3}+\frac{1}{8}+\frac{1}{24}+\frac{1}{35}+\ldots\right]$$

Harmonic Series

- Euler attacks the harmonic series and finds a link between the harmonic series and prime numbers:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\frac{2\cdot 3\cdot 5\cdot 7\cdots}{1\cdot 2\cdot 4\cdot 6\cdots}$$

- “the numerator on the right is the product of all the primes and the denominator is the product of all numbers one less than the primes.”

Cont.

- Keeping in mind that every whole number can be uniquely expressed as the product of primes, and with a substantial amount of work he deduces:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\frac{1}{1-\frac{1}{2}}\times\frac{1}{1-\frac{1}{3}}\times\frac{1}{1-\frac{1}{5}}\times\ldots=\frac{2\cdot 3\cdot 5\cdot 7\cdots}{1\cdot 2\cdot 4\cdot 6\cdots}$$

$$\sum_{k=1}^{\infty}\frac{1}{k}=\prod_{p}\frac{1}{1-\frac{1}{p}}$$

- The connection with Euclid is now made. The harmonic series is infinite, and therefore there are infinitely many primes.

Diverges

$$\sum_{k=1} \frac{1}{p_k}$$