MA 351 - Dr. A. Corso PRACTICE EXAM	No calculators or cheat sheets are allowed during the test.	The exam will be on Monday March 2 from 8:30-9:20 am in LAEB B222

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1. Consider the non homogeneous system of linear equations $A\mathbf{X} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & 4 \\ 2 & 1 & 1 & 0 & 3 \\ -1 & 3 & 1 & 5 & 4 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}.$$

   - Find all the solutions of the non homogeneous system $A\mathbf{X} = \mathbf{b}$.
   - Find all the solutions of the homogeneous system $A\mathbf{X} = \mathbf{0}$.
   - Observe that $\mathbf{X_1} = (2,-1,0,1,3)$ is a particular solution of $A\mathbf{X} = (12,12,12)$. Find all the solutions of the system $A\mathbf{X} = (12,12,12)$.

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2. Suppose that you have to solve the homogeneous system $A\mathbf{X} = \mathbf{0}$ with A a 4×5 matrix given by

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ -1 & -1 & 2 & -1 & 1 \\ 2 & 2 & -3 & 3 & 1 \\ 1 & 1 & 0 & -3 & 2 \end{bmatrix}.$$

   Find

   - all the solutions of the homogeneous system in terms of a basis of the nullspace of A;
   - a basis for the range of A

   given that

$$\mathtt{rref}(A|\mathbf{0}) = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 7/2 & 0 \\ 0 & 0 & 1 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

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3. Find a basis for the solution space of the homogeneous system of linear equations

$$\left\{ \begin{array}{rrrrcl} x & +y & -z & +2w & = & 0 \\ -x & +y & +4z & -3w & = & 0 \\ & 2y & +5z & +w & = & 0 \end{array} \right. .$$

   Check that $\mathbf{X_1} = (-5,2,-1,1)$ is a solution and write it in terms of that basis.

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4. Let V be the vector space consisting of all 2×2 matrices with real entries: $V = M_{2\times 2}(\mathbf{R})$. Show that the set W consisting of all 2×2 symmetric matrices

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a,b,c \text{ any real number} \right\}$$

   is a subspace of V.

   Find a basis for V. Find a basis for W. How many elements do they have?

5. Consider the following subset of $\mathbf{R}^5$

$$Z = \left\{ \begin{pmatrix} t \\ 3s+4t-2r \\ s \\ r \\ 3s-r+t \end{pmatrix} \; : \; s,t,r \text{ any real number} \right\}.$$

- Show that Z is closed under addition and under scalar multiplication: hence it is a subspace of $\mathbf{R}^5$. Can you exhibit a basis for Z?
- Show that Z is also the nullspace of some 2×5 matrix A. Find A.

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6. Show that the vectors $\mathbf{u} = (1,-1,1,0)$, $\mathbf{v} = (1,-2,0,1)$, and $\mathbf{w} = (2,0,1,1)$ are linearly independent vectors in $\mathbf{R}^4$. Let W be the subspace spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$: i.e., $W = \text{span}\langle \mathbf{u},\mathbf{v},\mathbf{w}\rangle$.

   Show that the vector $\mathbf{x} = (27,10,22,5)$ is in W and find α, β, and γ such that

$$\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}.$$

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7. Decide the dependence or independence of the following sets of vectors:

   (a) $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4$, and $\mathbf{v}_4 - \mathbf{v}_1$ for any vector $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$;

   (b) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$

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8. In the vector space V of all cubic polynomials $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, let S be the subset of polynomials with

$$\int_0^1 P(x)dx = 0.$$

   Verify that S is a subspace of V and find a basis for S.

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9. Let V be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Let

   (a) V_e be the subset of even functions, i.e., all functions f such that $f(-x) = f(x)$ for all $x \in \mathbf{R}$;

   (b) V_o be subset of odd functions, i.e., all functions f such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

   Prove that

   - V_e and V_o are both subspaces of V;
   - $V = V_e + V_o$, i.e., that every function $f \in V$ can be written as $f = f_1 + f_2$, where $f_1 \in V_e$ and $f_2 \in V_o$. (Hint: you may find useful to recall, for example, that $e^x = (e^x + e^{-x})/2 + (e^x - e^{-x})/2 = \cosh x + \sinh x$)

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9. Answer each of the following true or false. You must justify your statement.

(a) If some basis of a vector space V has four elements, then no set of three vectors in V is linearly independent.

(b) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n (the $n \times n$ matrix with 1's along the main diagonal and 0's elsewhere).

(c) If A is an $m \times n$ matrix and the row space of A has dimension r then the dimension of the null space of A is $m - r$.

(d) Five vectors could span the vector space of matrices $M_{2\times 2}(\mathbf{R})$.

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10. Determine all values of k such that the following system of equations

$$\begin{cases} x+y-z=2 \\ x+2y+z=3 \\ x+y+(k^2-5)z=k \end{cases}$$

has

(a) no solution;

(b) a unique solution;

(c) infinitely many solutions.
