MA 361 - 05/05/2003 FINAL EXAM	Spring 2003 A. Corso	Name: Answer Key

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; JUSTIFY YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
TOTAL	100	

1. (*i*) Compute the indicated product of cycles that are permutations of S_8

   * $(1,2)(4,7,8)(2,1)(7,2,8,1,5)$
   * $(1,4,5)(7,8)(2,5,7)$.

   (*ii*) Express the following permutation of S_8

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$$

   as product of disjoint cycles and then as product of transpositions.

   (*iii*) Consider the following permutations of S_6

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

   and compute

$$\tau\sigma, \qquad \tau^2\sigma, \qquad \sigma^{-1}\tau\sigma, \qquad \sigma^{100}, \qquad |\langle\tau^2\rangle|.$$

   (*iv*) Find the index of $\langle\sigma\rangle = (1,2,5,4)(2,3)$ in S_5.

All these are previous homework assignments and in previous tests---

Go check them----

pts: /20

2. (*i*) Draw the subgroup diagram for the subgroups of the group $\mathbb{Z}_{12}$.
   (*ii*) Find all cosets of the subgroup $\langle \overline{4} \rangle$ of $\mathbb{Z}_{12}$.
   (*iii*) Find the number of automorphisms of $\mathbb{Z}_{12}$.
   (*iv*) Find the order of $\overline{26} + \langle \overline{12} \rangle$ in $\mathbb{Z}_{60}/\langle \overline{12} \rangle$.

(i) We saw in an earlier exam that $\mathbb{Z}_{12}$ has the following diagram

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z}_{12} \arrow[dl, dash] \arrow[dr, dash] & & \\
\langle \overline{3} \rangle \arrow[dr, dash] & & \langle \overline{2} \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle \overline{6} \rangle \arrow[dr, dash] & & \langle \overline{4} \rangle = \{\overline{0}, \overline{4}, \overline{8}\} \arrow[dl, dash] \\
 & & \langle \overline{0} \rangle &
\end{tikzcd}$$

(ii) there are 4 cosets of $\langle \overline{4} \rangle$:

$\langle \overline{4} \rangle = \{\overline{0}, \overline{4}, \overline{8}\}$ $\quad \overline{1} + \langle \overline{4} \rangle = \{\overline{1}, \overline{5}, \overline{9}\}$

$\overline{2} + \langle \overline{4} \rangle = \{\overline{2}, \overline{6}, \overline{10}\}$ $\quad \overline{3} + \langle \overline{4} \rangle = \{\overline{3}, \overline{7}, \overline{11}\}$

(iii) $\mathbb{Z}_{12}$ has 4 generators: $\overline{1}, \overline{5}, \overline{7}, \overline{11}$.
Thus we have 4 distinct automorphism of $\mathbb{Z}_{12}$ since a generator has to be mapped to a generator

(iv) Notice that $\mathbb{Z}_{60}/\langle \overline{12} \rangle$ is a group of order $\frac{60}{|\langle \overline{12} \rangle|} = \frac{60}{5} = 12$ Thus the order of $\overline{26} + \langle \overline{12} \rangle$ must be a divisor of 12. Check that it is $\underline{6}$

pts: /20

3. Choose one of the following problems:

   (a) Determine whether the map
   $$\varphi\colon\ (M_2(\mathbb{R}),\cdot) \longrightarrow (\mathbb{R},\cdot),$$
   where $\varphi(A)=\det(A)$, is an isomorphism of binary structures.
   Explain.

   (b) Let F be the set of all functions f mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. Determine whether the map
   $$\varphi\colon\ (F,+) \longrightarrow (F,+),$$
   where $\varphi(f)(x)=\dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt$, is an isomorphism of binary structures.
   Explain.

(a) $\det(A\cdot B)=\det(A)\cdot\det(B)$ So φ is a homomorphism of groups. Notice that for $a\in\mathbb{R}$ then $A=\begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix}$ has $\det(A)=a$. Thus φ is surjective. However φ is <u>not injective</u> as different matrices can have the same determinant.

(b) φ is an isomorphism as $\varphi(f)(x)=$ $=\dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt=f(x)$ by the Fundamental Theorem of Calculus. I.e. $\varphi(f)=f$. I.e. φ is the <u>identity map</u>

pts: /15

4. Choose one of the following problems:

(*a*) Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

(*i*) Show that $*$ gives a binary operation on S.

(*ii*) Show that $(S, *)$ is a group.

(*b*) Let $\varphi\colon\ G \longrightarrow G'$ be an homomorphism of groups.
If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.

(a) (i) The issue is to show that for $a, b \neq -1$ then $a * b \neq -1$.
Suppose $a * b = a + b + ab = -1$. Then $a + b + ab + 1 = 0$ or $a(1+b) + b + 1 = 0$ or $(a+1)(b+1) = 0$, In $\mathbb{R}$ this implies that either $a = -1$ or $b = -1$. Thus $*$ is a binary operation.

(ii) • $(a * b) * c = (a + b + ab) * c = a + b + ab + c + (a + b + ab)c = a + (b + c + bc) + a(b + c + bc) = a * (b + c + bc) = a * (b * c)$ So $*$ is associative.

• $a * b = a$ for all $a \in \mathbb{R} \setminus \{-1\} \implies$ identity $\boxed{b = 0}$

$\cancel{a} + b + ab = \cancel{a}$ $\quad b(1+a) = 0 \implies$

• for every a, $\exists b$ $a + b + ab = 0$ inverse $\boxed{a^{-1} = b = \frac{-a}{1+a}}$

pts: /15

(b) It is a Theorem we have proved in class

5. Choose one of the following problems:
   (a) Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.
   (b) Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

(a) $a \sim b \iff ab^{-1} \in H$

(1) reflexive: $a \sim a$, This is true as

$aa^{-1} = e \in H$

(2) symmetric $a \sim b \iff ab^{-1} \in H$. But
H is a subgroup so $(ab^{-1})^{-1} \in H$
i.e. $(b^{-1})^{-1} a^{-1} \in H$ or $ba^{-1} \in H \iff$
$b \sim a$

(3) transitive: $a \sim b$ and $b \sim c \implies a \sim c$
$a \sim b \iff ab^{-1} \in H$; $b \sim c \iff bc^{-1} \in H$
But H is closed under $*$ so
$(ab^{-1})(bc^{-1}) = ac^{-1} \in H \iff a \sim c$

(b) Consider any $x \in G$ and $x^{-1}ax$. Notice
that $(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax) =$
$= x^{-1}a^2x = x^{-1}ex = x^{-1}x = e$.
But a is the unique of order 2

pts: /15

So $x^{-1}ax = a$ or $ax = ax$.

6. Choose one of the following problems:
   (*a*) A **torsion group** is a group all of whose elements have finite order. A group is **torsion free** if the identity is the only element of finite order.
   Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion free.
   (*b*) Show that the inner automorphisms of a group G form a normal subgroup of all automorphisms of G under compositions.

(a) $T=\{g\in G \mid g^n=e \text{ for some } n\in\mathbb{Z}^+\}$

Let $g\in T$ and $\tilde{g}$ any element of G.
We need to show that $\tilde{g}^{-1}g\tilde{g}\in T$. ~~As~~
$g^n=e$ (as $g\in T$) we have that

$$(\tilde{g}^{-1}g\tilde{g})^n=\underbrace{(\tilde{g}^{-1}g\tilde{g})(\tilde{g}^{-1}g\tilde{g})\cdots(\tilde{g}^{-1}g\tilde{g})}_{n \text{ times}}$$

$$=\tilde{g}^{-1}\underbrace{g^n}_{=e}\tilde{g}=\tilde{g}^{-1}\tilde{g}=e. \quad \therefore \tilde{g}^{-1}g\tilde{g}\in T.$$

Consider now $G/T=\{T, g_2T, \ldots, g_iT, \ldots\}$

Suppose $(g_iT)^k=e_{G/T}=T$ for some $k\in\mathbb{Z}^+$

observe that

$$(g_iT)^k=\underbrace{(g_iT)(g_iT)\cdots(g_iT)}_{k \text{ times}}=g_i^kT$$

If $g_i^kT=T \Rightarrow g_i^k\in T$. But then
$\exists\, n\in\mathbb{Z}^+$ such that $(g_i^k)^n=e_G$

pts: /15

$\Rightarrow g_i^{kn}=e_G \Rightarrow g_i\in T$.
So $g_iT=T \leftarrow$ the identity of G/T

(b) Let $g \in G$; the map

$i_g : G \longrightarrow G$, $i_g(x) = g^{-1} x g$

is called an inner automorphism

(exercise ; check that i_g is injective, surjective and $i_g(xy) = i_g(x) i_g(y)$!!)

$\mathrm{Inn}(G) = \{ i_g \mid g \in G \}$

$(i_{g_1} \circ i_{g_2})(x) = i_{g_1}(i_{g_2}(x)) = g_1^{-1}(g_2^{-1} x g_2) g_1 =$

$= (g_2 g_1)^{-1} x g_2 g_1 = i_{g_2 g_1}(x)$.

$\therefore$ $i_{g_1} \circ i_{g_2} \in \mathrm{Inn}(G)$.

Check that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G) = \{ \varphi : G \to G \mid \varphi \text{ isom} \}$

- i_e is the identity
- it is closed under composition as we saw above
- Check that $(i_g)^{-1} = i_{g^{-1}}$

Finally we want to check that

$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ is a normal subgroup.

Pick $i_g \in \mathrm{Inn}(G)$ and $\varphi \in \mathrm{Aut}(G)$

Want to check that $\varphi^{-1} \circ i_g \circ \varphi$ belongs to $\mathrm{Inn}(G)$.

But which kind of map $\varphi^{-1} \circ i_g \circ \varphi$ is?

Let's compute $(\varphi^{-1} \circ i_g \circ \varphi)(x) =$

$= \varphi^{-1}(i_g(\varphi(x))) = \varphi^{-1}(g^{-1}\varphi(x)\, g)$

$= \varphi^{-1}(g^{-1})\, \varphi^{-1}(\varphi(x))\, \varphi^{-1}(g) =$

$= [\varphi^{-1}(g)]^{-1}\, x\, \varphi^{-1}(g) = i_{\varphi^{-1}(g)}(x)$

$\therefore \quad \varphi^{-1} \circ i_g \circ \varphi = i_{\varphi^{-1}(g)} \in \mathrm{Inn}(G)$.

$\boxed{\#}$