MA 361 - 02/24/2010 FIRST MIDTERM	Spring 2010 A. Corso	Name: Answer Key

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; JUSTIFY YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	8	
2	7	
3	15	
4	10	
5	10	
Bonus	5	
TOTAL	55	/50

1. If $\mathcal{P} = \{\{1,3\},\{2\},\{4,5\}\}$, then $\mathcal{P}$ is a partition of the set $S = \{1,2,3,4,5\}$. For the corresponding equivalence relation $\sim$, which of the following are true or false?

(a) $4 \sim 5$	T ☑	F ☐
(b) $5 \sim 1$	T ☐	F ☑
(c) $1 \sim 2$	T ☐	F ☑
(d) $3 \sim 3$	T ☑	F ☐

pts: /8

2. Let $(G, *)$ be a group with identity e. If $a * b = e$ for some $a, b \in G$, show that $b * a = e$.

Suppose that $a * b = e$ then left multiply by b. We get $b * (a * b) = b * e$ By the associativity property and the fact that e is the identity we have

$(b * a) * b = b$ Now use cancellation on the right:

$b * a = e$.

pts: /7

3. (*a*) Write the arithmetic expression

$$\frac{i(3+i)}{2-4i}$$

in the form $a + ib$ for $a, b \in \mathbb{R}$.

(*b*) Find all solutions in $\mathbb{C}$ of the equation $z^3 = -8$.

(*c*) Write the complex number $z = 3 - 4i$ in the polar form.

(a) $\frac{i(3+i)}{2-4i} \cdot \frac{2+4i}{2+4i} = \frac{(-1+3i)(2+4i)}{2^2-(4i)^2} =$

$= \frac{-2-4i+6i-12}{4+16} = \frac{-14+2i}{20} = \boxed{-\frac{7}{10} + i\frac{1}{10}}$

(b) $z^3 = 8(-1) = 8(-1+i0)$

As $z = |z|e^{i\theta} \Rightarrow z^3 = |z|^3 \cdot e^{i3\theta} \Rightarrow$

$\Rightarrow |z|=2$ and $\cos(3\theta) = -1$ + $\sin(3\theta) = 0$.

$\Rightarrow 3\theta = \pi + 2k\pi \Rightarrow \underline{\theta = \frac{\pi}{3} + \frac{2}{3}k\pi}$ for

$k = 0, 1, 2 \Rightarrow \boxed{\theta_1 = \frac{\pi}{3}, \quad \theta_2 = \pi, \quad \theta_3 = \frac{5}{3}\pi}$

$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \underline{1+i\sqrt{3}}$; $z_2 = 2(-1+i\cdot 0) = \underline{-2}$

$z_3 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \underline{1+i\sqrt{3}}$

(c) $|z| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$

$\therefore z = 5\underbrace{\left(\frac{3}{5} - \frac{4}{5}i\right)}_{\text{has length one!}}$

pts: /15

4. Is $\varphi\colon (\mathbb{Z},+) \longrightarrow (\mathbb{Z},+)$, where $\varphi(n) = n - 3$ for all $n \in \mathbb{Z}$, an isomorphism? Explain.

In the case that φ is not an isomorphism, give the definition of a binary operation $*$ on $\mathbb{Z}$ such that φ is an isomorphism mapping $(\mathbb{Z},+)$ onto $(\mathbb{Z},*)$.

Let n and m such that

$n, m \in (\mathbb{Z}, *)$.

We have that $\underline{n * m} = \varphi(n+3) * \varphi(m+3)$

$= \varphi((n+3)+(m+3)) = \varphi(n+m+6)$

as we want φ to be an homomorphism;

$= (n+m+6) - 3 = \underline{n+m+3}$

Thus $\boxed{n * m = n+m+3}$

Observe that "e" $= -3$ is the identity for the operation $*$.

Notice that $\underline{\varphi(0)} = \underline{0 - 3 = -3}$.

pts: /10

5. Let G be a group and define $\varphi\colon\ G \longrightarrow G$, by setting $\varphi(g) = g^{-1}$ for all $g \in G$.

(i) Prove that φ is injective and surjective.

(ii) Suppose, in addition, that G is an Abelian group. Show that φ is an isomorphism of G onto itself.

(i) Let $g_1, g_2 \in G$ such that

$\varphi(g_1) = \varphi(g_2)$ i.e. $g_1^{-1} = g_2^{-1}$

This implies $g_1 * g_1^{-1} = g_1 * g_2^{-1}$ or

$e = g_1 * g_2^{-1}$. Now multiply by g_2. We get

$e * g_2 = (g_1 * g_2^{-1}) * g_2$ or $\underline{g_2 = g_1}$.

This proves the injectivity of φ.

Pick $g \in G$ then $g = (g^{-1})^{-1} =$

$= \varphi(g^{-1})$. Thus φ is surjective.

(ii) Finally for $g_1, g_2 \in G$.

$\varphi(g_1 * g_2) = (g_1 * g_2)^{-1} = g_2^{-1} * g_1^{-1} =$

$= \varphi(g_2) * \varphi(g_1) = \varphi(g_1) * \varphi(g_2)$

as G is commutativity with $*$.

pts: /10

Bonus. Complete the following table in such a way that $*$ is commutative and has an identity element and each element has an inverse.

$*$	W	X	Y	Z
W	Y	Z	W	X
X	Z	W	X	Y
Y	W	X	Y	Z
Z	X	Y	Z	W

identity= Y

X^{-1} = Z

Z^{-1}= X

W^{-1} = Y

pts: /5