MA 361 - 04/20/2012 THIRD MIDTERM (take home)	Spring 2012 A. Corso	Name: Answer Key

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; JUSTIFY YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
TOTAL	50	/50

1. Let $\varphi : G \longrightarrow G'$ be a group homomorphism.
Show that if $|G'|$ is finite, then $|\varphi(G)|$ is finite and is a divisor of $|G'|$.

We have proved in class that if $\varphi : G \longrightarrow G'$ is a homomorphism of groups, then $\varphi(G)$ is a subgroup of G'.

Since $|G'|$ is finite, by Lagrange Theorem we know that

$$|G'| = |\varphi(G)| \cdot [G' : \varphi(G)]$$

thus $|\varphi(G)|$ divides $|G'|$.

pts: /10

2. Find all left cosets of the subgroup $\{\rho_0, \mu_2\}$ of the group D_4 described by Table 8.12 (on page 80 of our textbook).

Let $H = \{\rho_0, \mu_2\}$

then $H = \boxed{\rho_0\{\rho_0, \mu_2\} = \mu_2\{\rho_0, \mu_2\}}$

$\boxed{\rho_1 H} = \rho_1\{\rho_0, \mu_2\} = \{\rho_1\rho_0, \rho_1\mu_2\}$

$= \{\rho_1, \delta_2\} = \boxed{\delta_2 H}$

$\boxed{\rho_2 H} = \{\rho_2\rho_0, \rho_2\mu_2\} = \{\rho_2, \mu_1\}$

$\boxed{= \mu_1 H}$

$\boxed{\rho_3 H} = \{\rho_3\rho_0, \rho_3\mu_2\} = \{\rho_3, \delta_1\}$

$\boxed{= \delta_1 H}$

Thus there are 4 left cosets

$[D_4 : H] = \frac{|D_4|}{|H|} = \frac{8}{2} = \boxed{4}$

pts: /10

3. Let S be any subset of a group G.

(a) Show that $H_S = \{x \in G \,|\, xs = sx \text{ for all } s \in S\}$ is a subgroup of G.

(b) In reference to part (a), the subgroup H_G is called the **center of G**.
Show that H_G is an abelian group.

(c) By analyzing Table 8.12 (on page 80 of our textbook), compute the center of the group D_4.

(a) Notice that $e_G s = s = s e_G$ for all $s \in S$

thus $e_G \in H_S$. I.e. H_S is non empty.

Let $x_1, x_2 \in H_S$ then $(x_1 x_2) s \overset{x_1}{=} (x_2 s) =$

$= x_1 (s x_2) = (x_1 s) x_2 = (s x_1) x_2 = s(x_1 x_2)$

Thus $x_1 x_2 \in H_S$. Finally, if $x \in H_S$

we have $xs = sx \Rightarrow s = x^{-1} s x \Rightarrow$

$s x^{-1} = x^{-1} s$ so that $x^{-1} \in H_S$.

$\therefore$ H_S is a subgroup

(b) H_G = center of $G = \{x \in G \,|\, xg = gx \text{ for all } g \in G\}$

If $x_1, x_2 \in H_G$ then $\boxed{x_1 x_2 = x_2 x_1}$
as both element are in G and also in H_G.

(c) $H_{D_4} = \{\rho_0, \rho_2\}$

pts: /10

4. Show that if H is a subgroup of index 2 in a finite group G, then every left coset of H is also a right coset of H.

Since $[G:H]=2$, this means that there are only 2 cosets of G and that G can be written as a disjoint union $G = \{H\} \dot\cup \{G \setminus H\}$ moreover

H and $G \setminus H$ have the same cardinality (which could be infinite)!

Thus for any $g \in G$ either $g \in H$ or $g \in G \setminus H$. If $g \in H$ then $gH = H = Hg$ (as H is closed under the operation). If $g \in G \setminus H$ then $gH = G \setminus H = Hg$.

Hence $gH = Hg$ for all $g \in G$.

$\therefore$ H is a normal subgroup

pts: /10

5. (*a*) Find the index of $\langle\overline{3}\rangle$ in the group $\mathbb{Z}_{24}$.

(*b*) Let $\sigma = (1\ 2\ 5\ 4)(2\ 3)$ in S_5. Find the index of $\langle\sigma\rangle$ in S_5.

(*c*) Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

(a) In $\mathbb{Z}_{24}$ $\langle\bar{3}\rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\}$

i.e. $|\langle\bar{3}\rangle| = 8$ that means by Lagrange theorem that $[\mathbb{Z}_{24} : \langle\bar{3}\rangle] = \dfrac{|\mathbb{Z}_{24}|}{|\langle\bar{3}\rangle|} = \dfrac{24}{8} = \boxed{3}$

(b) Observe that $\sigma = (1\ 2\ 5\ 4)(2\ 3) =$
$= (1\ 2\ 3\ 5\ 4)$ is a cycle of length 5.
Thus $\sigma^5 = id$; and 5 is the smallest such non-negative integer. Thus $|\langle\sigma\rangle| = |\sigma|$
$= 5$. thus $[S_5 : \langle\sigma\rangle] = \dfrac{|S_5|}{|\langle\sigma\rangle|} = \dfrac{120}{5} = \underline{\underline{24}}$

(c) Let $x \in gH$, i.e. $x = gh$ for some $h \in H$
But $gh = [(g^{-1})^{-1} h g^{-1}] g$. By assumption
$(g^{-1})^{-1} h g^{-1} \in H$ thus $gh = h_1 g$ for some $h_1 \in H$
So $gH \subseteq Hg$. Similarly
$Hg \subseteq gH$. Thus they are equal

pts: /10