Section 0
Homework Assignment

#5. Decide whether $\{n \in \mathbb{Z}^+ \mid n \text{ is a large number}\}$ is indeed a set (: is well defined).
Give an alternate description of each set.

Ans. It is not a well-defined set.

(Some may argue that no element of $\mathbb{Z}^+$ is large, because every element exceeds only a finite number of other elements but is exceeded by an infinite number of other elements. Such people might claim the answer should be $\emptyset$.)

#10. Decide whether $\{x \in \mathbb{Q} \mid x \text{ may be written with positive denominator less than } 4\}$ is indeed a set (: is well defined).
Give an alternate description of each set.

Ans. It is a well defined set.

"The set containing all numbers that are (positive, negative, or zero) integer multiples of 1, $1/2$, or $1/3$.

#16. List the elements of the power set of the given set and give the cardinality of the power set:

(a) $\emptyset$; (b) $\{a\}$; (c) $\{a,b\}$; (d) $\{a,b,c\}$

Ans. (a) $\mathcal{P}(\emptyset)$: $\emptyset$; cardinality $1 = 2^0$.

(b) $\mathcal{P}(\{a\})$: $\emptyset, \{a\}$; cardinality $2 = 2^1$.

(c) $\mathcal{P}(\{a,b\})$: $\emptyset, \{a\}, \{b\}, \{a,b\}$;

cardinality $4 = 2^2$

(d) $\mathcal{P}(\{a,b,c\})$: $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\},$

$\{a,c\}, \{b,c\}, \{a,b,c\}$; cardinality

$8 = 2^3$.

#17. Let A be a finite set, and let $|A| = s$. Based on the preceding exercise, make a conjecture about the value of $|\mathcal{P}(A)|$. Then try to prove your conjecture.

Ans. Conjecture: $|\mathcal{P}(A)| = 2^s$ when $|A| = s$.

Proof The number of subsets of a set A depends only on the cardinality of A, not on what the elements of A actually are. Suppose $B = \{1, 2, 3, \ldots, s-1\}$ and $A = \{1, 2, 3, \ldots, s\}$. Then A has all the elements of B plus the one additional element s. All subsets of B are also subsets of A; these are precisely the subsets of A that do not contain s, so the number of subsets of A not containing s is $|\mathcal{P}(B)|$. Any other subset of A must contain s, and removal of the s would produce a subset of B. Thus the number of subsets of A containing s is also $|\mathcal{P}(B)|$. Because every subset of A either contains s or does not contain s (but not both), we see that the number of subsets of A is $2|\mathcal{P}(B)|$.

We have seen that if A has one more element than B, then $|\mathcal{P}(A)| = 2|\mathcal{P}(B)|$. Now, $|\mathcal{P}(\emptyset)| = 1$

and so by induction:

$$|\mathcal{P}(A)| = 2 \cdot 2^{s-1} = 2^s.$$

∎

#36. Let $n \in \mathbb{Z}^+$ and let $\sim$ be defined on $\mathbb{Z}$ by $r \sim s$ if and only if $r-s$ is divisible by n, that is, if and only if $r-s = nq$ for some $q \in \mathbb{Z}$.

(a) Show that $\sim$ is an equivalence relation on $\mathbb{Z}$ (: "The congruence modulo n").

(b) Show that, when restricted to the subset $\mathbb{Z}^+$ of $\mathbb{Z}$, this $\sim$ is the equivalence relation of Example 0.20.

(c) The cells of this partition of $\mathbb{Z}$ are the residue classes modulo n in $\mathbb{Z}$. Write the residue classes modulo

$$n = 2,\ 3,\ 5 \quad \text{in } \mathbb{Z}.$$

Ans.

(a) Let h, k and m be any integers. We need to verify the following properties:

- reflexive: $h - h = n \cdot 0 = 0$, so $h \sim h$

- symmetric: If $h \sim k$ so that $h-k = nq$ for some $q \in \mathbb{Z}$, then $k-h = n(-q)$ so $k \sim h$.

Transitive: If $h \sim k$ and $k \sim m$ then $h-k = nq_1$ and $k-m = nq_2$ for some $q_1, q_2 \in \mathbb{Z}$. But then

$$h - m = (h-k) + (k-m) = nq_1 + nq_2 = n(q_1+q_2)$$

with $q_1+q_2 \in \mathbb{Z}$. So $h \sim m$.

(b) Let $h, k \in \mathbb{Z}^+$. In the sense of this exercise, $h \sim k$ if and only if $h-k = nq$ for some $q \in \mathbb{Z}$.

In the sense of Example 0.20,

$h \equiv k \pmod{n}$ if and only if h and k have the same remainder when divided by n. Write

$h = nq_1 + r$ and $k = nq_2 + r$

So $h - k = n(q_1 - q_2)$; so $h-k$ is a multiple of n. So $h \sim k$.

(c) if $n = 2$:

$\overline{0} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$ $\overline{1} = \{\ldots, -3, -1, 1, 3, \ldots\}$

if $\underline{n = 3}$

$\bar{0} = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

$\bar{1} = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots$

$\bar{2} = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$

if $\underline{n = 5}$

$\bar{0} = \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\}$

$\bar{1} = \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\}$

$\bar{2} = \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\}$

$\bar{3} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

$\bar{4} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$