Section 10

Homework Assignment

#1. Find all cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$.

Ans:

There are 4 cosets:

$4\mathbb{Z}$, $1+4\mathbb{Z}$, $2+4\mathbb{Z}$ and $3+4\mathbb{Z}$.

#3. Find all cosets of the subgroup $\langle \bar{2} \rangle$ of $\mathbb{Z}_{12}$.

Ans:

There are 2 cosets: $\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$

and $\bar{1} + \langle \bar{2} \rangle = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \overline{11}\}$.

#4. Find all cosets of the subgroup $\langle \bar{4} \rangle$ of $\mathbb{Z}_{12}$.

Ans:

There are 4 cosets: $\langle \bar{4} \rangle = \{\bar{0}, \bar{4}, \bar{8}\}$,

$\bar{1} + \langle \bar{4} \rangle = \{\bar{1}, \bar{5}, \bar{9}\}$, $\bar{2} + \langle \bar{4} \rangle = \{\bar{2}, \bar{6}, \overline{10}\}$

and $\bar{3} + \langle \bar{4} \rangle = \{\bar{3}, \bar{7}, \overline{11}\}$.

\# 12. Find the index of $\langle \bar{3} \rangle$ in the group $\mathbb{Z}_{24}$.

<u>Ans</u>:

$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\}$

$\therefore |\langle \bar{3} \rangle| = 8$. Thus the index of $\langle \bar{3} \rangle$ in $\mathbb{Z}_{24}$ is $\dfrac{|\mathbb{Z}_{24}|}{|\langle \bar{3} \rangle|} = \dfrac{24}{8} = 3$.

\# 13. Find the index of $\langle \mu_1 \rangle$ in the group S_3.

<u>Answer</u>: $|\langle \mu_1 \rangle| = 2$ so that the index of $\langle \mu_1 \rangle$ in S_3 is $:= \dfrac{|S_3|}{|\langle \mu_1 \rangle|} = \dfrac{6}{2} = 3$.

\# 14. Find the index of $\langle \mu_2 \rangle$ in the group D_4.

<u>Ans</u>: $\langle \mu_2 \rangle = \{id, \mu_2\}$ so that $|\langle \mu_2 \rangle| = 2$ and the index of $\langle \mu_2 \rangle$ in D_4 is $= \dfrac{8}{2} = 4$.

#15. Let $\sigma = (1,2,5,4)(2,3)$ in S_5. Find the index of $\langle\sigma\rangle$ in S_5.

Ans:

Observe that $\sigma = (1,2,3,5,4)$ so that $|\langle\sigma\rangle| = 5$. Thus the index of $\langle\sigma\rangle$ in S_5 is $= \frac{|S_5|}{5} = \frac{120}{5} = 24$.

#16. Let $\mu = (1,2,4,5)(3,6)$ in S_6. Find the index of $\langle\mu\rangle$ in S_6.

Ans:

Observe that μ generates a cyclic subgroup of S_6 of order 4 (the cycles are disjoint) so its index (the number of left cosets) is $6!/4 = \frac{720}{4} = 180$.

#28. Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

Ans:

We show that $gH = Hg$ by showing that each coset is a subset of the other.

Let $gh \in gH$ where $g \in G$ and $h \in H$.

Then
$$gh = ghg^{-1}g$$
$$= \left[(g^{-1})^{-1} h g^{-1}\right] g$$

is in Hg because $(g^{-1})^{-1} h g^{-1}$ is in H by hypothesis. Thus gH is a subset of Hg.

Now let $hg \in Hg$, when $g \in G$ and $h \in H$. Then

$$hg = gg^{-1}hg = g(g^{-1}hg)$$

is in gH because $g^{-1}hg$ is in H by hypothesis. Thus Hg is a subset of gH as well.

Thus $Hg = gH$. ∎

#29. Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$.

Ans:

Let $h \in H$ and $g \in G$. By hypothesis $Hg = gH$. Thus $hg = gh_1$ for some $h_1 \in H$. Thus $g^{-1}hg = h_1$, showing that

$$g^{-1}hg \in H$$

for all $g \in G$ and $h \in H$. ∎

#39. Show that if H is a subgroup of index 2 in a finite group G, then every left coset of H is also a right coset of H.

Ans:

The partition of G into left cosets of H must be

$$H \quad \text{and} \quad G \setminus H = \{g \in G \mid g \notin H\}$$

because G has finite order and H must have half as many elements as G. For the same reason, this must be the partition into right cosets of H. Thus every left coset is also a right coset. ∎