Section 13
Homework Assignment

#16. Compute $\operatorname{Ker}\varphi$ for $\varphi: S_3 \longrightarrow \mathbb{Z}_2$ defined by

$$\varphi(\sigma) = \begin{cases} \bar{0} & \text{if } \sigma \text{ is even} \\ \bar{1} & \text{if } \sigma \text{ is odd} \end{cases}$$

Ans:

$\operatorname{Ker}\varphi$ consists of the even permutations, so $\operatorname{Ker}\varphi = A_3$.

#17. Compute $\operatorname{Ker}\varphi$ and $\varphi(25)$ for $\varphi: \mathbb{Z} \to \mathbb{Z}_7$ such that $\varphi(\bar{1}) = \bar{4}$.

Ans:

As $\bar{4}$ has order 7 in $\mathbb{Z}_7$, we have that φ has kernel equal to $7\mathbb{Z}$. Thus

$$\begin{aligned} \varphi(25) &= \varphi(21+4) = \varphi(21) + \varphi(4) = \bar{0} + \varphi(4) \\ &= \varphi(1+1+1+1) = \varphi(1) + \varphi(1) + \varphi(1) + \varphi(1) \\ &= \bar{2} \end{aligned}$$

#18. Compute Ker φ and $\varphi(18)$ for $\varphi: \mathbb{Z} \longrightarrow \mathbb{Z}_{10}$ such that $\varphi(1) = \bar{6}$.

Ans:

As $\bar{6}$ has order 5 in $\mathbb{Z}_{10}$ we have that Ker $\varphi = 5\mathbb{Z}$. Thus

$$\varphi(18) = \varphi(15+3) = \varphi(15) + \varphi(3) = \bar{0} + \varphi(3)$$
$$= \varphi(1+1+1) = \varphi(1) + \varphi(1) + \varphi(1)$$
$$= \bar{6} + \bar{6} + \bar{6} = \bar{8}$$

#28. Let G be a group, and let $g \in G$. Let $\varphi_g : G \longrightarrow G$ be defined by $\varphi_g(x) = gx$ for $x \in G$. For which $g \in G$ is φ_g a homomorphism?

Ans:

As a homomorphism sends the identity to the identity, we must have

$$\varphi_g(e) = e$$

i.e. $\quad ge = e \quad \therefore \quad g = e.$

I.e., there is only one homomorphism: φ_e.

#29. Let G be a group, and let $g \in G$. Let $\varphi_g : G \longrightarrow G$ be defined by $\varphi_g(x) = gxg^{-1}$ for $x \in G$. For which $g \in G$ is φ_g a homomorphism?

Ans:

We have $\varphi_g(xy) = g(xy)g^{-1} = gxg^{-1}gyg^{-1}$

$$= \varphi_g(x)\,\varphi_g(y)$$

for all $x, y \in G$. Thus φ_g is a homomorphism for all $g \in G$.

#48. The sign of an even permutation is $+1$ and the sign of an odd permutation is -1. Observe that the map

$$sgn_n : S_n \longrightarrow \{1, -1\}$$

defined by $sgn_n(\sigma) =$ sign of σ is a homomorphism of S_n onto the multiplicative group $\{1, -1\}$. What is the kernel?

Ans: We see that $Ker(sgn_n) = A_n$. The multiplicative group $\{-1, 1\}$ is isomorphic to the group $\mathbb{Z}_2$, and if we rename $1 \in \{1, -1\}$ with $\bar{0}$ and we rename -1 with $\bar{1}$, then this becomes the homomorphism of Example 13.3.

#49. Show that if G, G', and G'' are groups and if $\varphi : G \longrightarrow G'$ and $\gamma : G' \longrightarrow G''$ are homomorphisms, then the composite map

$$\gamma \circ \varphi : G \longrightarrow G''$$

is a homomorphism.

Ans:

Let $a, b \in G$. In the composite function we have:

$$\begin{aligned}
(\gamma \circ \varphi)(ab) &= \gamma(\varphi(a,b)) && \text{def. of comp.} \\
&\overset{!}{=} \gamma(\varphi(a)\varphi(b)) && \varphi \text{ is a homom.} \\
&\overset{!}{=} \gamma(\varphi(a))\,\gamma(\varphi(b)) && \gamma \text{ is a homom.} \\
&\overset{!}{=} (\gamma \circ \varphi)(a)\,(\gamma \circ \varphi)(b) && \text{def. of comp.}
\end{aligned}$$

This shows that $\gamma \circ \varphi$ is indeed a homomorphism.

#50. Let $\varphi : G \longrightarrow H$ be a group homomorphism. Show that $\varphi[G]$ is abelian if and only if for all $x, y \in G$ we have $xyx^{-1}y^{-1} \in \operatorname{Ker} \varphi$.

<u>Ans:</u>

For all $\varphi(x)$ and $\varphi(y) \in \varphi[G]$ (i.e. for all $x, y \in G$) we have that

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$$

$$\Longleftrightarrow$$

$$\varphi(xy) = \varphi(yx)$$

$$\Longleftrightarrow$$

$$\varphi(xy)\,\varphi(yx)^{-1} = e'$$

$$\Longleftrightarrow$$

$$\varphi(xy)\varphi((yx)^{-1}) = e'$$

$$\Longleftrightarrow$$

$$\varphi(xy(yx)^{-1}) = e'$$

$$\Longleftrightarrow$$

$$xyx^{-1}y^{-1} \in \operatorname{Ker} \varphi.$$

■

#52. Let $\varphi : G \longrightarrow G'$ be a homomorphism with Kernel H and let $a \in G$. Prove the set equality

$$\{x \in G \mid \varphi(x) = \varphi(a)\} = Ha$$

Ans:

If $ha = x \in Ha$ we have that

$$\varphi(x) = \varphi(ha) = \varphi(h)\,\varphi(a) = e'\varphi(a) = \varphi(a).$$

Thus $Ha \subseteq \{x \in G \mid \varphi(x) = \varphi(a)\}$.

Conversely if $x \in G$ is such that $\varphi(x) = \varphi(a)$ we have that

$\iff \varphi(x)\,\varphi(a)^{-1} = e' \iff \varphi(x a^{-1}) = e'$

$\iff x a^{-1} \in \operatorname{Ker}\varphi = H$.

Thus $x a^{-1} = h \in H$ for some h, and $x = ha \in Ha$. As desired. ∎