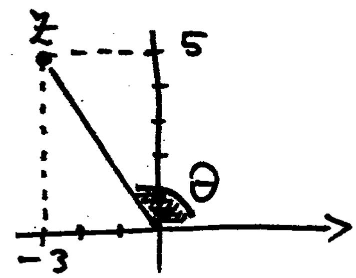
Section 1
Homework assignment

#3. $i^{23} = (i^2)^{11} \cdot i = (-1)^{11} \cdot i = (-1) \cdot i = i$

#15. Write $z = -3 + 5i$ in the polar form.




<u>Ans.</u> $|z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

$\therefore\ z = \sqrt{34}\,(\cos\theta + i\sin\theta)$

when $\theta$ is the angle in the picture.

Also, from the picture we get that

$$\cos\theta = \frac{-3}{\sqrt{34}} \qquad \sin\theta = \frac{5}{\sqrt{34}}$$

#19. Find all solutions in $\mathbb{C}$ of the equation

$$z^3 = -27i$$

<u>Ans</u>: If $z = |z|\,(\cos\theta + i\sin\theta)$ we have

$$z^3 = |z|^3\left(\cos(3\theta) + i\sin(3\theta)\right) \underset{\substack{\uparrow \\ \text{WANT}}}{=} 3^3\left(0 + (-1)i\right)$$

$\therefore\ |z|^3 = 3^3$ and $\cos(3\theta) = 0$ and $\sin(3\theta) = -1$.

$\therefore\ |z| = 3$ while $3\theta = \frac{3}{2}\pi + 2\pi k \quad k \in \mathbb{Z}$

$\therefore\ |z| = 3$ and $\theta = \frac{\pi}{2} + \frac{2}{3}\pi k \quad k = 0, 1, 2$

This yields the following choices for $\theta$

$\theta = \pi/2$ , $\theta = \frac{7}{6}\pi$ , $\theta = \frac{11}{6}\pi$

Eventually we get the 3 solutions

$$z_1 = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 3i$$

$$z_2 = 3\left(\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

$$z_3 = 3\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

#20. Find all the solutions in $\mathbb{C}$ of the equation $z^6 = 1$

<u>Ans</u>. If $z = |z|(\cos\theta + i\sin\theta)$ , then

$$z^6 = |z|^6(\cos(6\theta) + i\sin(6\theta))$$

$\therefore z^6 = 1 \implies |z|^6 = 1$ , $\cos(6\theta) = 1$ $\sin(6\theta) = 0$

or $|z| = 1$ and $6\theta = 0 + 2\pi k$ $k \in \mathbb{Z}$

But since we want the solution with $0 \le \theta < 2\pi$ we have:

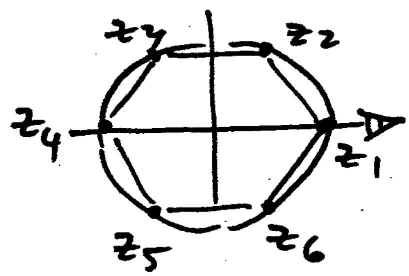
$$\theta = \frac{\pi}{3}k \qquad k = 0, 1, 2, 3, 4, 5$$

$\therefore \theta = 0, \pi/3, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{5}{3}\pi$

Hence $z_1 = 1$, $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_4 = -1$, $z_5 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $z_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

are all the solutions of $z^6 = 1$.




they are the vertices of the regular hexagon!

#33. Find all solutions $x$ of the equation

$$x +_{12} x = 2 \quad \text{in } \mathbb{Z}_{12}$$

Ans An obvious solution is $x = 1$. Otherwise, we need to have $x + x = 2 + 12$, so $x = 7$ will also work.

Check that none of the remaining 10 elements of $\mathbb{Z}_{12}$ is a solution.

#35. Example 1.15 asserts that there is an isomorphism of $U_8$ with $\mathbb{Z}_8$ in which $\zeta = e^{i\frac{\pi}{4}} \longleftrightarrow \bar{5}$ and $\zeta^2 \longleftrightarrow \bar{2}$. Find the element of $\mathbb{Z}_8$ that corresponds to each of the remaining six elements

$\zeta^m$ in $U_8$ for $m = 0, 3, 4, 5, 6, 7$.

<u>Ans.</u> $\zeta^0 \longleftrightarrow \bar{0}$, $\zeta \longleftrightarrow \bar{5}$, $\zeta^2 = \zeta \cdot \zeta \longleftrightarrow$

$\longleftrightarrow \bar{5} +_8 \bar{5} = \bar{2}$, $\zeta^3 = \zeta \cdot \zeta \cdot \zeta \longleftrightarrow \bar{5} +_8 \bar{5} +_8 \bar{5}$

$= \bar{7}$, $\zeta^4 \longleftrightarrow \bar{4}$, $\zeta^5 \longleftrightarrow \bar{1}$

$\zeta^6 \longleftrightarrow \bar{6}$, $\zeta^7 \longleftrightarrow \bar{3}$.

#41. Recall the power series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + x^3/3! + \cdots$$

$$\sin x = x - x^3/3! + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - x^2/2! + x^4/4! - \cdots$$

from calculus. Derive Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ formally from these three series expansions.

<u>Ans</u>: $$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta. \qquad \blacksquare$$