Section 2
Homework Assignment

Let $*$ be the binary operation on $S=\{a,b,c,d,e\}$ defined by means of the Table

$*$	a	b	c	d	e
a	a	b	c	b	d
b	b	c	a	e	c
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

#1. Compute $b*d$, $c*c$, and $[(a*c)*e]*a$.

Ans. $b*d=e$; $c*c=b$

$$[(a*c)*e]*a = [c*e]*a = a*a=a$$

#2. Compute $(a*b)*c$ and $a*(b*c)$. Can you say on the basis of this computation whether $*$ is associative?

Ans. $(a*b)*c = b*c = a$

$$a*(b*c) = a*a = a$$

We can't tell whether $*$ is associative.

We need to check all other triplets.

#3. Compute $(b * d) * c$ and $b * (d * c)$. Can you say on the basis of this computation whether $*$ is associative?

Ans:

$$(b * d) * c = e * c = a$$

$$b * (d * c) = b * b = c$$

$\neq$

Thus the operation is not associative. We found a counterexample.

#4. Is $*$ commutative? why?

Ans: No : $e * b = b$ whereas $b * e = c$

#5. Complete Table 2.27 so as to define a commutative binary operation $*$ on $S = \{a, b, c, d\}$

$*$	a	b	c	d
a	a	b	c	...
b	b	d	...	c
c	c	a	d	b
d	d	...	...	a

Ans: the table must be symmetric with respect to the main diagonal

$*$	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

#7. Determine whether the binary operation $*$ defined on $\mathbb{Z}$ by letting $a*b = a-b$ is commutative and whether $*$ is associative.

<u>Ans</u>: $a*b = a-b$ whereas

$b*a = b-a = -(a-b)$

$\therefore *$ is <u>not</u> commutative.

It is anticommutative.

$$a*(b*c) = a-(b*c) = a-(b-c)$$

$$(a*b)*c = (a*b)-c = a-b-c$$

But $a-b-c \neq a-b-c$.

$\therefore *$ is <u>not</u> associative.

#8. Determine whether the binary operation $*$ defined on $\mathbb{Q}$ by letting $a*b = ab+1$ is commutative and whether $*$ is associative.

<u>Ans</u>: $a*b = ab+1 \underset{\uparrow}{=} ba+1 = b*a$

product in $\mathbb{Q}$ is commutative

$\therefore *$ is commutative.

$$a*(b*c) = a(b*c)+1 = a(bc+1)+1$$

$$= abc + a + 1$$

on the other hand

$$(a * b) * c = (a*b)c + 1 = (ab+1)c + 1$$
$$= abc + c + 1$$

$\therefore$ $*$ is <u>not</u> associative.

#14. "A binary operation $*$ is commutative if and only if $a*b = b*a$."

<u>Ans.</u> This statement is incorrect. Mention should be made of the underlying set for $*$ and the universal quantifier, <u>for all</u>, should appear.

"A binary operation $*$ on a set S is commutative if and only if $a*b =$ $= b*a$ for all $a, b \in S$."

#15. "A binary operation $*$ on a set S is associative if and only if, for all $a, b, c \in S$, we have $(b*c)*a =$ $= b*(c*a)$."

<u>Ans.</u> The definition is correct.

#16. "A subset H of a set S is closed under a binary operation $*$ on S if and only if $a*b \in H$ for all $a, b \in S$."

<u>Ans</u>. It is incorrect. Replace the final S by H.

#17. Determine whether the definition of $*$ on $\mathbb{Z}^+$ by letting $a*b = a-b$ does give a binary operation on the set.

<u>Ans</u>: It is not a binary operation. Infact $1*1 = 0 \notin \mathbb{Z}^+$.

So $\mathbb{Z}^+$ is <u>not</u> closed under $*$.

#23. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under

(a) matrix multiplication?

(b) matrix addition?

<u>Ans</u>. Yes:

(a) $$\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - a_2b_1 \\ b_1a_2 + a_1b_2 & -b_1b_2 + a_1a_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + a_2b_1) \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{bmatrix}$$

so it is closed under matrix multiplication

(b) $\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & -(b_1+b_2) \\ b_1+b_2 & a_1+a_2 \end{bmatrix}$

$\therefore$ so it is closed under matrix addition.

#24.

(a) If $*$ is a binary operation on any set S, then $a * a = a$ for all $a \in S$.

False

(b) If $*$ is any commutative binary operation on any set S, then $a*(b*c) = (b*c)*a$ for all $a, b, c \in S$.

True

(c) If $*$ is any associative binary operation on any set S, then $a*(b*c) = (b*c)*a$ for all $a, b, c \in S$.

False

(d) The only binary operations of any importance are those defined on sets of numbers.

False

(e) A binary operation $*$ on a set S is commutative if there exist $a, b \in S$ such that $a*b = b*a$.

<u>False</u>

(f) Every binary operation defined on a set having exactly one element is both commutative and associative.

<u>True</u>

(g) A binary operation on a set S assigns at least one element of S to each ordered pair of elements of S.

<u>True</u>

(h) A binary operation on a set S assigns at most one element of S to each ordered pair of elements of S.

<u>True</u>

(i) A binary operation on a set S assigns exactly one element of S to each ordered pair of elements of S.

<u>True</u>

(j) A binary operation on a set S may assign more than one element of S to some ordered pair of elements of S. <u>False</u>