Section 3
Homework Assignment

#1. What three things must we check to determine whether a function $\varphi : S \longrightarrow S'$ is an isomorphism of a binary structure $(S, *)$ with $(S', *')$?

Ans. 1.) φ must be one-one

2.) φ must be onto

3.) φ must be an homomorphism, i.e.

$$\varphi(a * b) = \varphi(a) *' \varphi(b) \text{ for all } a, b \in S.$$

#2. Is $\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}, +)$, where $\varphi(n) = -n$ for all $n \in \mathbb{Z}$, an isomorphism?

Ans. Yes: 1.) one-one
if $\varphi(n_1) = \varphi(n_2)$ then
$-n_1 = -n_2$ or $n_1 = n_2$.

2.) onto: if $m \in \mathbb{Z}$ (codomain)
then $\varphi(?) = m$ for some "?"

Pick $\varphi(-m) = m$.

3.) $\varphi(n+m) = -(n+m) = -n - m$
$= \varphi(n) + \varphi(m)$ for all $n, m \in \mathbb{Z}$

#3. Is $\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}, +)$, where $\varphi(n) = 2n$ for all $n \in \mathbb{Z}$, an isomorphism?

<u>Ans.</u> <u>No</u> : It is not onto. If I pick $5 \in \mathbb{Z}$ (codomain), can I find "?" such that $\varphi(?) = 5$? <u>Nope</u>, because I should choose $? = 5/2 \notin \mathbb{Z}$.

<u>Notice</u> that φ is one-one : $\varphi(n_1) = \varphi(n_2)$ or $2n_1 = 2n_2$ implies $n_1 = n_2$.
Also, $\varphi(n+m) = 2(n+m) = 2n + 2m$
$= \varphi(n) + \varphi(m)$.

#4. Is $\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}, +)$, where $\varphi(n) = n+1$ for all $n \in \mathbb{Z}$, an isomorphism?

<u>Ans.</u> <u>No</u> : It is not an homomorphism.

Indeed, $\varphi(n+m) = n + m + 1$

$$\neq$$

whereas $\varphi(n) + \varphi(m) = (n+1) + (m+1)$
$\overset{!}{=} n + m + 2$

However, φ is one-one and onto!
Check it.

#5. Is $\varphi : (\mathbb{Q},+) \longrightarrow (\mathbb{Q},+)$, where $\varphi(x) = x/2$ for all $x \in \mathbb{Q}$, an isomorphism?

Ans. Yes:

one-one: if $\varphi(x_1) = \varphi(x_2)$ or $\frac{x_1}{2} = \frac{x_2}{2}$ we conclude that $x_1 = x_2$.

onto: if $y \in \mathbb{Q}$ (codomain), can we find "?" $\in \mathbb{Q}$ (domain) such that $\varphi(?) = y$? Yes, Pick "?" $= 2y$, so

$$\varphi(2y) = \frac{2y}{2} = y.$$

Homomorphism: $\varphi(x_1 + x_2) = \frac{x_1+x_2}{2} = \frac{x_1}{2} + \frac{x_2}{2}$

for all $x_1, x_2 \in \mathbb{Q}$. $= \varphi(x_1) + \varphi(x_2)$,

#6. Is $\varphi : (\mathbb{Q},\cdot) \longrightarrow (\mathbb{Q},\cdot)$, where $\varphi(x) = x^2$, an isomorphism?

Ans. No: It is not one-one nor onto.

$\varphi(x) = \varphi(-x) = x^2$ (not one-one)

$\varphi(\text{"?"}) = -1$ (or any negative number)?

nope, I can't find any "?" that works.

<u>However</u>, φ is an homomorphism:

$$\varphi(x_1 x_2) = (x_1 x_2)^2 = x_1^2 x_2^2 = \varphi(x_1)\varphi(x_2),$$

for all $x_1, x_2 \in \mathbb{Q}$.

#7. Is $\varphi : (\mathbb{R}, \cdot) \longrightarrow (\mathbb{R}, \cdot)$, when $\varphi(x) = x^3$ for all $x \in \mathbb{R}$, an isomorphism?

<u>Ans.</u> <u>Yes</u>: <u>one-one</u>: if $\varphi(x_1) = \varphi(x_2)$ then $x_1^3 = x_2^3$ implies $x_1 = x_2$.

<u>onto</u>: if $y \in \mathbb{R}$ (codomain) then $\varphi(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y$. So φ is onto

<u>homomorphism</u>: $\varphi(x_1 x_2) = (x_1 x_2)^3 = x_1^3 x_2^3 =$
for all $x_1, x_2 \in \mathbb{R}$. $= \varphi(x_1)\varphi(x_2)$

#8. Is $\varphi : (M_2(\mathbb{R}), \cdot) \longrightarrow (\mathbb{R}, \cdot)$, when $\varphi(A) = \det(A)$ for all $A \in M_2(\mathbb{R})$, an isomorphism?

<u>Ans.</u> <u>No</u>: it's not one-one. For example $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ have both determinant zero.

<u>However</u>, φ is both <u>onto</u> and a <u>homomorphism</u>. Why?

Let F be the set of all functions $f: \mathbb{R} \to \mathbb{R}$ that have derivatives of all orders.

#11. Is $\varphi: (F,+) \longrightarrow (F,+)$, where $\varphi(f) = f'$, an isomorphism?

Ans. No: two functions that differ by a constant have the same derivative:

$$\varphi(x^2+1) = 2x = \varphi(x^2)$$. So it is not one-one.

However, φ is onto and a homomorphism.

Given $f \in F$ (codomain) then

$$\varphi\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

So φ is onto. Also $\varphi(f+g) = (f+g)' =$

$= f' + g' = \varphi(f) + \varphi(g)$, so it is a homomorphism.

#12. Is $\varphi: (F,+) \longrightarrow (\mathbb{R},+)$, where $\varphi(f) = f'(0)$, an isomorphism?

Ans.: No: It is not one-one

For example: $\varphi(\sin x) = \cos(0) = 1$
$\varphi(x) = 1$

Again, φ is onto and a homomorphism.

#13. Is $\varphi : (F,+) \longrightarrow (F,+)$, where $\varphi(f)(x) = \int_0^x f(t)\,dt$, an isomorphism?

Ans.: No, it is not onto. In fact if you pick $x+1 \in F$ (codomain) we cannot find any $f \in F$ (domain) such that $\varphi(f)(x) = x+1$. In fact for $x=0$ $\varphi(f)(0) = \int_0^0 f(t)\,dt = 0$ while $0+1 = 1$.

However, φ is one-one and a homomorphism Why?

#14. Is $\varphi : (F,+) \longrightarrow (F,+)$, where $\varphi(f)(x) = \frac{d}{dx}\int_0^x f(t)\,dt$, an isomorphism?

Ans. Yes: $\varphi(f)(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$ by the Fundamental Theorem of Calculus.

Thus φ is the identity function, which is clearly one-one, onto and an homomorphism.

#15. Is $\varphi : (F, \cdot) \longrightarrow (F, \cdot)$, where $\varphi(f)(x) = x f(x)$, an isomorphism?

Ans. Note that $\varphi(f)(0) = 0 \cdot f(0) = 0$.

No. Thus there is no way that I can find $f \in F$ so that $\varphi(f) = 1$ (i.e. the function constantly equal to 1).

#16. The map $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $\varphi(n) = n+1$ for $n \in \mathbb{Z}$ is one to one and onto $\mathbb{Z}$. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that φ is an isomorphism mapping

(a) $(\mathbb{Z}, +)$ onto $(\mathbb{Z}, *)$

(b) $(\mathbb{Z}, *)$ onto $(\mathbb{Z}, +)$

Ans. (a) For all $m, n \in \mathbb{Z}$

$$m * n = \varphi(m-1) * \varphi(n-1) = \varphi\big((m-1) + (n-1)\big) =$$

$$= \varphi(m+n-2) = (m+n-2)+1 = m+n-1$$

$\therefore \quad m*n = m+n-1 \quad$ for all $m, n \in \mathbb{Z}$

Also "1" is the identity element.

(b) We want $\varphi : (\mathbb{Z}, *) \longrightarrow (\mathbb{Z}, +)$ to be an isomorphism.

$$\underbrace{\varphi(m*n)}_{= m*n+1} = \varphi(m) + \varphi(n) = (m+1)+(n+1) = m+n+2$$

$\therefore \quad m*n+1 = m+n+2$

$\therefore \quad m*n = m+n+1 \quad$ for all $m, n \in \mathbb{Z}$.

Also, "-1" is the identity element.

#18. The map $\varphi : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined by $\varphi(x) = 3x-1$ for $x \in \mathbb{Q}$ is one to one and onto $\mathbb{Q}$. Give the definition of a binary operation $*$ on $\mathbb{Q}$ such that φ is an isomorphism mapping:

(a) $(\mathbb{Q}, +)$ onto $(\mathbb{Q}, *)$

(b) $(\mathbb{Q}, *)$ onto $(\mathbb{Q}, +)$

Ans. (a) As before

$$x * y = \varphi\left(\frac{x+1}{3}\right) * \varphi\left(\frac{y+1}{3}\right) =$$

$$= \varphi\left(\frac{x+1}{3} + \frac{y+1}{3}\right) = \varphi\left(\frac{x+y+2}{3}\right)$$

$$= 3\left(\frac{x+y+2}{3}\right) - 1 = x+y+1$$

And the identity is "-1".

(b) As before: $\varphi : (\mathbb{Q}, *) \longrightarrow (\mathbb{Q}, +)$

$$\varphi(x*y) = \varphi(x) + \varphi(y) = (3x-1)+(3y-1)$$

$$= 3(x*y) - 1 \qquad\qquad = 3x+3y-2$$

So: $3(x*y) - 1 = 3(x+y) - 2$

$$x * y = x + y - 1/3$$

And the identity element is "$1/3$".

#33. Let $H = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$.

Exercise 23 of Section 2 shows that H is closed under both matrix addition and matrix multiplication.

(a) Show that $(\mathbb{C}, +) \cong (H, +)$ and (b) $(\mathbb{C}, \cdot) \cong (H, \cdot)$

(a) Define $\varphi : (\mathbb{C},+) \longrightarrow (H,+)$ by

$$\varphi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{for } a,b \in \mathbb{R}$$

Clearly φ is one-one and onto H.

We have

$$\varphi((a+ib)+(c+id)) = \varphi((a+c)+i(b+d)) =$$

$$= \begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$= \varphi(a+ib) + \varphi(c+id)$$

so φ is an isomophism.

(b) Define φ in the same way.
Now:

$$\varphi((a+ib)(c+id)) = (ac-bd+i(ad+bc))$$

$$= \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-dd \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

$$= \varphi(a+ib)\,\varphi(c+id).$$

So, φ is an isomophism. ▦