Elementary Modern Algebra I, MA 361 – Spring 2012
Homework set # 5 (Section 4) (due on **February 24** (**Friday**), **2012**)

1. (#8 on page 45) We can also consider multiplication $\cdot_n$ modulo n in $\mathbb{Z}_n$. For example, $5 \cdot_7 6 = 2$ in $\mathbb{Z}_7$ because $5 \cdot 6 = 30 = 4(7) + 2$. The set $\{1, 3, 5, 7\}$ with multiplication $\cdot_8$ modulo 8 is a group. Give the table for this group

2. (#19 on page 46) Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab.$$

a. Show that $*$ gives a binary operation on S.
b. Show that $(S, *)$ is a group.
c. Find the solution of the equation $2 * x * 3 = 7$ in S.

3. (#28 on page 48) From our intuitive grasp of the notion of isomorphic groups, it should be clear that if $\varphi : G \longrightarrow G'$ is a group isomorphism, then $\varphi(e)$ is the identity e' of G'. Recall that Theorem 3.14 gave a proof of this for isomorphic binary structures $(S, *)$ and $(S', *')$. Of course, this covers the case of groups.

It should also be intuitively clear that if a and a^{-1} are inverse pairs in G, then $\varphi(a)$ and $\varphi(a^{-1})$ are inverse pairs in G', that is,

$$\varphi(a)^{-1} = \varphi(a^{-1}).$$

Give a careful proof of this for a skeptic who can't see the forest from all the trees.

4. (#31 on page 48) If $*$ is a binary operation on a set S, an element x of S is an **idempotent for** $*$ if $x * x = x$. Prove that a group has exactly one idempotent element. (You may use any theorems proved so far in the text.)

5. (#32 on page 48) Show that every group G with identity e such that $x * x = e$ for all $x \in G$ is abelian. [*Hint:* Consider $(a * b) * (a * b)$.]

6. (#34 on page 49) Let G be a group with a finite number of elements. Show that for any $a \in G$, there exists an $n \in \mathbb{Z}^+$ such that $a^n = e$. See Exercise 33 for the meaning of a^n. [*Hint:* Consider $e, a, a^2, a^3, \ldots, a^m$, where m is the number of elements in G, and use cancellation laws.]

7. (#41 on page 49) Let G be a group and let g be one fixed element of G. Show that the map i_g, such that

$$i_g(x) = gxg^{-1}$$

for $x \in G$, is an isomorphism of G with itself.