Section 5

Homework Assignment

#8. Determine whether the set of invertible $n \times n$ matrices with real number entries and determinant 2 is a subgroup of $GL(n, \mathbb{R})$.

Ans: $H = \{ A \in GL(n, \mathbb{R}) \mid \det A = 2 \}$ is **not** a subgroup of $GL(n, \mathbb{R})$.

Indeed, if $A, B \in H$ then $\det(AB) =$
$= \det(A) \det(B) = 2 \cdot 2 = 4$ $\therefore$ $AB \notin H$.

Even $I_n \notin H$ as $\det(I_n) = 1$.

#12. Determine whether the set of invertible $n \times n$ matrices with real number entries and determinant -1 or 1 is a subgroup of $GL(n, \mathbb{R})$.

Ans: Yes. The set is closed under product as $\det(AB) = \det(A)\det(B)$ and if $\det(A) = +1$ or -1 and $\det(B) = +1$ or -1 then the same is true for $\det(AB)$.

The set is non empty as $\det(I_n) =$.

Finally, it is closed under taking the inverse as
$$\det(A^{-1}) = 1/\det(A).$$

#13. Determine whether the set of all invertible $n \times n$ matrices with real entries and such that $A^T A = I_n$ is a subgroup of $GL(n, \mathbb{R})$.

Ans: Yes. Suppose that $A^T A = I_n$ and $B^T B = I_n$. Then we have that $(AB)^T AB = B^T(A^T A)B = B^T I_n B = B^T B = I_n$, so the set of these matrices is closed under multiplication. Since $I_n^T = I_n$ and $I_n I_n = I_n$, the set contains the identity. For each A in the set, the equation $A^T A = I_n$ shows that A has an inverse A^T. The equation $(A^T)^T A^T = A A^T = I_n$ shows that A^T is in the given set. Thus we have a subgroup.

#15. Let F be the set of all real-valued functions with domain $\mathbb{R}$ and let $\tilde{F}$ be the subset of F consisting of those functions that have a nonzero value at every point in $\mathbb{R}$.
Determine whether the subset of all $f \in F$ such that $f(1) = 0$ with the induced operation is

(a) a subgroup of F under addition

(b) a subgroup of $\tilde{F}$ under multiplication.

Ans: (a) Yes. Let $H = \{ f \in F \mid f(1) = 0 \}$.

We have that if $f, g \in H$ then
$(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$
so that $f + g \in H$.
Moreover, the function identically equally to 0 is the identity of H.
Finally, for any $f \in H$ also its additive inverse $-f$ is in H
as $(-f)(1) = -f(1) = -0 = 0$.
Thus H is a subgroup of F.

(b) No. In fact H is not even contained in $\tilde{F}$.

#22. Describe all the elements in the cyclic subgroup of $GL(2, \mathbb{R})$ generated by the 2×2 matrix

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Ans: Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and observe that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus

$$\langle A \rangle = \underset{\text{generated by } A}{\text{cyclic subgroup}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

#27 Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by $\overline{3}$.

<u>Ans:</u> $\langle \overline{3} \rangle = \{ \overline{0}, \overline{3}, \underbrace{\overline{3}+\overline{3}}_{= \overline{2}}, \underbrace{\overline{3}+\overline{3}+\overline{3}}_{= \overline{1}}, \underbrace{\overline{3}+\overline{3}+\overline{3}+\overline{3}}_{= \overline{0}} \}$

etc...

$$\therefore = \{ \overline{0}, \overline{1}, \overline{2}, \overline{3} \}$$

$$\therefore |\langle \overline{3} \rangle| = 4.$$

#28 Find the order of the cyclic subgroup of V_4 generated by c.

<u>Ans:</u> We recall that the multiplication table of V_4 is:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\therefore \langle c \rangle = \{ e, c \}$ as $c^2 = e$, $c^3 = c$,

etc....

#36. (a) Write the group table of $\mathbb{Z}_6$.
(b) Compute the subgroups $\langle \bar{0} \rangle$, $\langle \bar{1} \rangle$, $\langle \bar{2} \rangle$, $\langle \bar{3} \rangle$, $\langle \bar{4} \rangle$, and $\langle \bar{5} \rangle$ of the group $\mathbb{Z}_6$.
(c) Which elements are generators for the group $\mathbb{Z}_6$ of part (a)?
(d) Give the subgroup diagram for the part (b) subgroups of $\mathbb{Z}_6$.

Ans:
(a)

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	5	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{5}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

(b) $\langle \bar{0} \rangle = \{\bar{0}\}$, $\langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$,

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}\}$ $\quad \langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{2}, \bar{4}\}$ $\quad \langle \bar{5} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

(c) The previous calculation shows that $\bar{1}$ and $\bar{5}$ are the generators of $\mathbb{Z}_6$.

(d) Moreover, the lattice subgroup of $\mathbb{Z}_6$ is

$$\begin{tikzcd}[column sep=small]
 & \langle \bar{1} \rangle = \langle \bar{5} \rangle = \mathbb{Z}_6 \arrow[dl, dash] \arrow[ddr, dash] & & \text{-------} 6 \text{ elts} \\
\langle 2 \rangle = \langle 4 \rangle \arrow[ddr, dash] & & & \text{-------} 3 \text{ elts} \\
 & & \langle \bar{3} \rangle \arrow[dl, dash] & \cdots 2 \text{ elts} \\
 & \langle \bar{0} \rangle & & \text{-------} 1 \text{ elt.}
\end{tikzcd}$$

#41. Let $\varphi: G \longrightarrow G'$ be an isomorphism of a group $(G, *)$ with a group $(G', *')$. Show that if H is a subgroup of G then $\varphi(H) = \{ \varphi(h) \mid h \in H \}$ is a subgroup of G'.
That is, an isomorphism carries subgroups into subgroups.

Ans: Let $\varphi(a), \varphi(b) \in \varphi(H)$. Now $a * b \in H$ as H is a subgroup of G.
Thus $\varphi(a) *' \varphi(b) \underset{\text{as } \varphi \text{ is a homomorphism.}}{=} \varphi(a * b) \in \varphi(H)$

This shows that $\varphi(H)$ is closed under product.

Observe that $e' = \varphi(e) \in \varphi(H)$, where e is the identity of G and H.

Finally, let $a \in H$ so that $\varphi(a) \in \varphi(H)$. Since H is a subgroup of G we also have that $a^{-1} \in H$. Thus

$$e' = \varphi(e) = \varphi(a * a^{-1}) = \varphi(a) *' \varphi(a^{-1})$$

shows that $\varphi(a)^{-1} = \varphi(a^{-1}) \in \varphi(H)$.

Thus $\varphi(H)$ is a subgroup of G'.

#42. Let $\varphi: G \longrightarrow G'$ be an isomorphism of a group $(G, *)$ with a group $(G', *')$. Show that if G is cyclic then G' is also cyclic.

<u>Ans.</u> Let a be a generator of G.
We <u>claim</u> that $\varphi(a)$ is a generator of G'.

Since $\varphi(a) \in G'$ we clearly have that $\langle \varphi(a) \rangle$ is contained in G'. Thus we need to show that

$$G' \subseteq \langle \varphi(a) \rangle$$

Pick $b' \in G'$. Since φ is onto there exists $b \in G$ such that

$$\varphi(b) = b'.$$

But $G = \langle a \rangle$ is a cyclic group generated by a. Thus there exists $n \in \mathbb{Z}$ such that $b = a^n$. Thus

$$b' = \varphi(b) = \varphi(a^n) = \varphi(\underbrace{a * \cdots * a}_{n\text{-times}})$$

as φ is an homomorphism

$$= \underbrace{\varphi(a) *' \varphi(a) *' \cdots *' \varphi(a)}_{n\text{-times}}$$

$$= \varphi(a)^n.$$

Thus $G' \subseteq \langle \varphi(a) \rangle$, which shows that $G' = \langle \varphi(a) \rangle$ is cyclic. ∎

#43. Show that if H and K are subgroups of an abelian group then

$$HK = \{ hk \mid h \in H, k \in K \}$$

is a subgroup of G.

<u>Ans:</u> Let $h_1 k_1$ and $h_2 k_2$ be in HK. Then

$$(h_1 k_1)(h_2 k_2) = \underset{\underset{H}{\Uparrow}}{(h_1 h_2)} \underset{\underset{K}{\Uparrow}}{(k_1 k_2)} \in HK$$

since G is abelian and H and K are subgroups. This shows that

HK is closed under product.

Moreover $e \in H, K$ implies that $e = \underset{H}{e} \cdot \underset{K}{e} \in HK$. So HK contains the identity element of G.

Finally, let $h \in H$ and $k \in K$. Since H and K are subgroups of G we have that $h^{-1} \in H$ and $k^{-1} \in K$. We show that $h^{-1}k^{-1} \in HK$ is the inverse of hk.

Indeed, $(h^{-1}k^{-1})hk = h^{-1}hk^{-1}k = ee = e$

$$(hk)(h^{-1}k^{-1}) = hh^{-1}kk^{-1} = ee = e$$

As G is abelian. Thus HK is a subgroup of G. ∎

#45. (MORE COMPACT SUBGROUP CRITERIA)

Show that a non-empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

<u>Ans.</u> Let H be a subgroup of G (with the usual definition). Then for $a, b \in H$, we have that $b^{-1} \in H$

and $ab^{-1} \in H$ because H must be closed under the induced operation.

Conversely, suppose that H is non-empty and $ab^{-1} \in H$ for all $a, b \in H$.

Let $a \in H^{\neq \phi}$. Then taking $b = a$ we see that $e = aa^{-1} \in H$. Taking $a = e$ and $b = a$, we see that $ea^{-1} = a^{-1} \in H$. Thus H contains the identity element e and the inverse of each element. For the closure of the operation, note that for $a, b \in H$ we also have $a, b^{-1} \in H$ and thus $a(b^{-1})^{-1} = ab \in H$.

#47. Prove that if G is an abelian group, written multiplicatively, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G.

Ans. Let $H = \{x \in G \mid x^2 = e\}$

Closure: if $a, b \in H$ then

$$(ab)^2 = abab \underset{\substack{\uparrow \\ G \text{ is abelian}}}{=} aabb = a^2b^2 = ee = e$$

$\therefore ab \in H$.

Identity e is in H : infact $e^2 = e.e = e$.

Inverse : if $a \in H$ then $a^2 = a \cdot a = e$.

Thus $a^{-1} = a$. So $a^{-1} \in H$.

This shows that H is a subgroup of G. $\blacksquare$

#50. Let a non-empty finite subset H of a group G be closed under the binary operation of G. Show that H is a subgroup of G.

Ans: Let $a \in H$ and let H have n elements. Then the elements

$$a, a^2, a^3, \ldots, a^{n+1} \text{ are all in } H$$

(because H is closed under the operation), and cannot all be different.

$\therefore$ $a^i = a^j$ for some $i < j$.

Then multiplication by a^{-i} shows

$$e = a^{j-i} \in H$$

Also $a^{-1} = a^{j-i-1} \in H$. Thus H is a subgroup of G. ∎

#51. Let G be a group and let a be one fixed element of G. Show that

$$H_a = \{ x \in G \mid xa = ax \}$$

is a subgroup of G.

<u>Ans</u>. Let $x, y \in H_a$, i.e. $xa = ax$ and $ya = ay$. We want to show that $xy \in H_a$. But

$$(xy)a = x(ya) = x(ay) = (xa)y = (ax)y$$
$$\overset{!}{=} a(xy) \quad \therefore \; xy \in H_a.$$

Obviously, $ea = a = ae$ so that $e \in H_a$.

Finally, from $xa = ax$ we get

$$xax^{-1} = a\underbrace{xx^{-1}}_{e} = a \quad \text{so} \quad xax^{-1} = a.$$

But then $\underbrace{x^{-1}x}_{e}ax^{-1} = x^{-1}a$ so that

$$ax^{-1} = x^{-1}a \quad \therefore \quad x^{-1} \in H_a.$$

$\therefore$ H_a is a subgroup of G.

#53. Let H be a subgroup of a group G. For $a, b \in G$ let $a \sim b$ iff $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

Ans. Reflexive property : $a \sim a$ as

$aa^{-1} = e \in H$.

Symmetric property : if $a \sim b$ then $ab^{-1} \in H$. But H is closed under inverse so $(ab^{-1})^{-1} \in H$ or $ba^{-1} \in H$, which says that $b \sim a$.

Transitive property : if $a \sim b$ an $b \sim c$ we have that $ab^{-1}, bc^{-1} \in H$. But then $(ab^{-1})(bc^{-1}) = ac^{-1} \in H$ as H is closed under product. $\therefore a \sim c$.

#54. Show that if $H \le G$ and $K \le G$ then $H \cap G$ is also a subgroup of G.

Suppose $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H and K are subgroups then $ab \in H$ and $ab \in K$. Thus $ab \in H \cap K$. This shows that $H \cap K$ is closed under product.

Since $e \in H$ and $e \in K$ we have that the identity $e \in H \cap K$.

Finally if $a \in H \cap K$ then $a \in H, K$ and $a^{-1} \in H, K$ since they are both subgroups. Thus $a^{-1} \in H \cap K$. ∎

So $H \cap K$ is closed under inverse.

#55. Prove that every cyclic group is abelian

<u>Ans</u>

Suppose $G = \langle a \rangle$. Pick $x, y \in G$

i.e. $x = a^n$ and $y = a^m$ for $n, m \in \mathbb{Z}$.

Then $xy = a^n \cdot a^m = a^{n+m} = a^{m+n} = a^m a^n = yx$.

which shows that G is abelian.

#57. Show that a group with no proper nontrivial subgroups is cyclic.

Ans: If $G=\{e\}$, then G is of course cyclic. If $G \neq \{e\}$, then let $a \in G$, $a \neq e$. We know that $\langle a \rangle$ is a subgroup of G and $\langle a \rangle \neq \{e\}$. Because G has no proper nontrivial subgroups, we must have $\langle a \rangle = G$, so G is indeed cyclic. ∎