Section 9
Homework Assignment

#1. Find all orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$$

Ans:

$\{1, 5, 2\}$, $\{3\}$ and $\{4, 6\}$.

#2. Find all orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$$

Ans:

$\{1, 5, 8, 7\}$, $\{2, 6, 3\}$ and $\{4\}$.

#3. Find all orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$$

Ans:

$\{1, 2, 3, 5, 4\}$ and $\{6\}$ and $\{7, 8\}$.

#4. Let $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\sigma(n) = n+1$. Find all the orbits of σ.

Ans: the is only one orbit, namely $\mathbb{Z}$.

#5. Let $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\sigma(n) = n+2$. Find all the orbits of σ.

Ans:

There are 2 orbits: $\{2n \mid n \in \mathbb{Z}\}$ and $\{2n+1 \mid n \in \mathbb{Z}\}$.

#6. Let $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\sigma(n) = n-3$. Find all orbits of σ.

Ans:

There are 3 orbits:

$\{3n \mid n \in \mathbb{Z}\}$, $\{3n+1 \mid n \in \mathbb{Z}\}$ and

$\{3n+2 \mid n \in \mathbb{Z}\}$.

#7. Compute the indicated product of cycles: $(1,4,5)(7,8)(2,5,7)$ that are elements of S_8.

Ans: $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}$$

#8. Compute the indicated product of cycles: $(1,3,2,7)(4,8,6)$ that are element of S_8.

Ans:
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 2 & 8 & 5 & 4 & 1 & 6 \end{pmatrix}$$

#9. Compute the indicated product of cycles: $(1,2)(4,7,8)(2,1)(7,2,8,1,5)$ that are elements of S_8.

Ans:
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 7 & 8 & 6 & 2 & 1 \end{pmatrix}$$

#10. Express the permutation

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

as a product of disjoint cycles, and then as a product of transposition.

Ans:
$$\sigma_1 = (1,8)(3,6,4)(5,7)$$
$$= (1,8)(3,4)(3,6)(5,7)$$

#11. Express the permutation

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$

as a product of disjoint cycles,

and then as a product of transpositions.

<u>Ans:</u>

$$\sigma_2 = (1,3,4)(2,6)(5,8,7)$$
$$= (1,4)(1,3)(2,6)(5,7)(5,8)$$

#12. Express the permutation

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$$

as a product of disjoint cycles, and then as a product of transpositions.

<u>Ans:</u>

$$\sigma_3 = (1,3,4,7,8,6,5,2)$$
$$= (1,2)(1,5)(1,6)(1,8)(1,7)(1,4)(1,3)$$

#13. Recall that element a of a group G with identity element e has order $r > 0$ if $a^r = e$ and no smaller positive power of a is the identity. Consider the group S_8.

(a) What is the order of the cycle $(1,4,5,7)$?

(b) State a theorem suggested by part (a).

(c) What is the order of $\sigma = (4,5)(2,3,7)$? of $\tau = (1,4)(3,5,7,8)$?

(d) Find the order of each of the permutations given in Exercises 10 through 12 by looking at its decomposition into a product of disjoint cycles.

(e) State a theorem suggested by parts (c) and (d).

Ans:

(a) if $\sigma = (1,4,5,7)$ we have that $\sigma^4 = e$ whereas $\sigma^3 \neq e$.
$\therefore$ $|\sigma| = 4$.

(b) Theorem: A cycle of length n has order n.

(d) $|\sigma_1| = 6$, $|\sigma_2| = 6$, $|\sigma_3| = 8$.

(c) $|\sigma| = 6$ and $|\tau| = 4$.

(e) Theorem: The order of a permutation expressed as a product of disjoint cycles is the least common multiple of the lengths of the cycles.

#34. Show that if σ is a cycle of odd length, then σ^2 is a cycle.

Ans:

Suppose $\sigma = (a_1, a_2, a_3, \ldots, a_m)$, where m is odd. But then

$$\sigma^2 = (a_1, a_3, a_5, \ldots, a_m, a_2, a_4, a_6, \ldots, a_{m-1})$$

which is again a cycle.

#36. Let G be a group and let a be a fixed element of G. Show that the map $\lambda_a : G \longrightarrow G$, given by $\lambda_a(g) = ag$ for $g \in G$, is a permutation of the set G.

Ans:

We must show that λ_a is one-to-one and onto G. Suppose that $\lambda_a(g_1) = \lambda_a(g_2)$. Then $ag_1 = ag_2$. The group cancellation property then yields $g_1 = g_2$, so λ_a is one-to-one. Let $g \in G$. Then $\lambda_a(a^{-1}g) = a(a^{-1}g) = g$, so λ_a is onto G. ▦

#39. Show that S_n is generated by $\{(1,2), (1,2,3,\ldots,n)\}$.

Ans:

We show that

$$(1,2,3,\ldots,n)^r (1,2)(1,2,3,\ldots,n)^{n-r} = (r+1, r+2)$$

for $0 \le r \le n-1$ [: where for $r = n-1$ we obtain the transposition $(n,1)$!!!].

To see this, note that any number not mapped into 1 or 2 by $(1,2,3,\ldots,n)^{n-r}$ is left fixed by the given product. For $r = i$, we see that $(1,2,3,\ldots,n)^{n-i}$ maps $i+1$ into 1, which is then mapped into 2 by $(1,2)$, which is mapped into $i+2$ by $(1,2,3, —, n)^i$. Also $(1,2,3, —, n)^{n-i}$ maps $i+2$ mod n into 2, which is then mapped into 1 by $(1,2)$, which is mapped into $i+1$ by $(1,2,3, —, n)^i$.

Let (i,j) be any transposition, written with $i<j$. We easily compute that

$$(i,j) = (i,i+1)(i+1,i+2)\cdots(j-2,j-1)(j-1,j)\cdot$$
$$\cdot(j-2,j-1)\cdots(i+1,i+2)(i,i+1)$$

By Corollary 9.12, every permutation can be written as a product of transpositions, which we now see can each be written as a product of the special transpositions $(1,2)$, $(2,3)$, $\cdots$, $(n-1,n)$, $(n,1)$ and we have shown that these in turn can be expressed as products of $(1,2)$ and $(1,2,3,\text{—},n)$. This completes the proof. ∎