The due date is on October 3 (Wednesday), 2001.

1. Prove that a group $G$ cannot have a subgroup $H$ with $|H| = n - 1$, where $n = |G| > 2$.
2. Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of a group $G$. Prove that $\bigcup_{i=0}^{\infty} H_i$ is a subgroup of $G$.

3. $\star$ Let $G$ be an abelian group. Prove that $H = \{ g \in G \mid |g| < \infty \}$ is a subgroup of $G$, called the torsion subgroup of $G$.

4. Let $H$ be a subgroup of the group $G$.

   $\star$ Show that $H \leq N_G(H)$.

   $\star$ Show that $H \leq C_G(H)$ if and only if $H$ is abelian.

5. Let $H$ be a subgroup of order 2 in $G$. Show that $N_G(H) = C_G(H)$. Deduce that if $N_G(H) = G$ then $H \leq Z(G)$.

6. Prove that the subgroup generated by any two distinct elements of order 2 in $S_3$ is all of $S_3$.

7. Draw the lattice subgroup of: $\mathbb{Z}/16\mathbb{Z}$ and $\mathbb{Z}/24\mathbb{Z}$.

8. Let $G = \{1, a, b, c\}$ be a group of order 4. Show that either $G \cong \mathbb{Z}/4\mathbb{Z}$ or $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

9. Let $G = \langle x \rangle$ be a cyclic subgroup of order $n$. Our goal is to show that the group $(\text{Aut}(G), \circ)$ of automorphisms of $G$ is an abelian group of order $\varphi(n)$, where $\varphi$ is Euler’s function.

   For each integer $a$ define the map

   $\sigma_a: G \longrightarrow G, \quad x \mapsto \sigma_a(x) = x^a$.

   $\star$ Prove that $\sigma_a$ is an automorphism of $G$ if and only if $(a, n) = 1$.

   $\star$ Prove that $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$.

   $\star$ Prove that every automorphism of $G$ is equal to $\sigma_a$ for some $a$.

   $\star$ Prove that $\sigma_a \circ \sigma_b = \sigma_{ab}$.

   Deduce that the map

   $\theta: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \text{Aut}(G), \quad \overline{a} \mapsto \theta(\overline{a}) = \sigma_a$

   is an isomorphism.

10. Let $G$ be a finite group of order $n$. Use Lagrange’s Theorem to show that the map

    $\gamma: G \longrightarrow G, \quad g \mapsto \gamma(g) = g^k$

    is surjective for any integer $k$ relatively prime to $n$. That is, for such integer $k$ any element $g \in G$ has a $k^{th}$ root in $G$. 

1