Homework set # 4

The due date is on October 19 (Friday), 2001.

1. Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = \{1\}$.

2. Let H and K be normal subgroups of a group G with $H \cap K = \{1\}$. Show that

- $\star$ $hk = kh$ for every $h \in H$ and $k \in K$.
- $\star$ HK is a subgroup of G with $HK \cong H \times K$.
- $\star$ Give an example of a group G with two such subgroups H and K.

3. Let H and K be normal subgroups of G such that $G = HK$. Prove that $G/(H \cap K) \cong (G/H) \times (G/K)$. In particular, if $H \cap K = \{1\}$ one has that $G \cong (G/H) \times (G/K)$.

4. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^{\times}$ to prove *Fermat's Little Theorem*:

If p is a prime then $\quad a^p \equiv a \mod p \quad$ for all $a \in \mathbb{Z}$.

5. Let $H \leq G$ and let $g \in G$. Prove that if the right coset Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$.

6. Let G be a group. For any $g \in G$ define the map

$$f_g : G \longrightarrow G,\ x \mapsto f_g(x) = gxg^{-1},$$

called *inner automorphism* (or *conjugation*).

- $\star$ Verify that f_g is an automorphism.
- $\star$ Show that the map $g \mapsto f_g$ is a homomorphism of G into $\mathrm{Aut}(G)$ with kernel $Z(G)$.
- $\star$ Conclude that $\mathrm{Inn}(G) = \{f_g \,|\, g \in G\}$ is a subgroup of $\mathrm{Aut}(G)$ with $\mathrm{Inn}(G) \cong G/Z(G)$.
- $\star$ Verify that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *group of outer automorphisms*.

7. Prove that if G is an abelian group of order pq, where p and q are distinct primes, then G is cyclic.

8. Let $N \lhd G$ and $M \leq G$ be subgroups of a group G. If both M and N are solvable then so is MN.

9. Prove the Jordan-Hölder Theorem.

(*Hint*: use induction and the second Isomorphism Theorem.)

10. Let H be a subgroup of a group G of finite index. Show that there exists a normal subgroup N of G of finite index with $N \subset H$.