Linear Algebra, MA 565 – Fall 2011
Homework set # 4 (due on **September 27** (**Wednesday**), **2011**)

13. (#11 on page 84) Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Prove that the range and the null space of T are disjoint, i.e., have only the zero vector in common.

14. (#7 on page 86) Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \mapsto UTU^{-1}$ is an isomorphism of $\mathcal{L}(V, V)$ onto $\mathcal{L}(W, W)$.

15. Let U, V, W be finite dimensional vectors spaces over k. Suppose that

$$0 \to U \xrightarrow{T_1} V \xrightarrow{T_2} W \to 0$$

is a short exact sequence, that is, T_1 and T_2 are linear transformations such that T_1 is injective, T_2 is surjective and $\operatorname{im}(T_1) = \ker(T_2)$. Repeat the same proof we did in class for the Rank-Nullity Theorem to show that

$$\dim(V) = \dim(U) + \dim(W).$$

In particular, observe that such proof establishes the existence of a linear transformation $S: W \longrightarrow V$ such that $T_2 \circ S$ is the identity on W. Moreover, $V = T_1(U) \oplus S(W) \cong U \oplus W$.

16. Let W be a subspace of a finite dimensional vector space V. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ be a basis of W. Extend this basis to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_l, \mathbf{v}_{l+1}, \ldots, \mathbf{v}_n\}$ of V. Show that $\{\mathbf{v}_{l+1} + W, \ldots, \mathbf{v}_n + W\}$ is a basis of V/W.

17. (**Strong Form of the First Isomorphism Theorem for Vector Spaces**) Let $T: V \longrightarrow W$ be a linear transformation of vector spaces over k. Suppose that $U \subseteq \ker(T)$ is a subspace. Show that there exists a unique linear transformation $\overline{T}: V/U \longrightarrow \operatorname{im}(T)$ such that $\overline{T} \circ \pi = T$, where $\pi: V \longrightarrow V/U$. Show that $\overline{T}$ is surjective and $\ker \overline{T} = \ker(T)/U$. Conclude that $(V/U)/(\ker(T)/U) \cong \operatorname{im}(T)$.

18. (**Third Isomorphism Theorem for Vector Spaces**) Let V be a vector space over k and let $U \subseteq W \subseteq V$ be subspaces. Then

$$(V/U)/(W/U) \cong V/W.$$

If in addition $\dim(V/U)$ is finite show that

$$\dim(V/U) = \dim(V/W) + \dim(W/U).$$

Remark: Another way of rephrasing the statements in Problems **16.**, **17.**, and **18.** is that

$$0 \to W \longrightarrow V \longrightarrow V/W \to 0,$$

$$0 \to \ker(T)/U \longrightarrow V/U \xrightarrow{\overline{T}} \operatorname{im}(T) \to 0,$$

and

$$0 \to W/U \longrightarrow V/U \longrightarrow V/W \to 0$$

are short exact sequences of vectors space, with the obvious definition of the unlabeled maps.