Determining If Two Graphs Are Isomorphic

Given two graphs, it is often really hard to tell if they ARE isomorphic, but usually easier to see if they ARE NOT isomorphic. Here is our first idea to help tell if two graphs are isomorphic.

**Theorem (Isomorphic Graphs Theorem 1)**

Suppose we have two graphs. In the first graph there are $v_1$ vertices and $e_1$ edges. In the second graph there are $v_2$ vertices and $e_2$ edges. Then in order for the two graphs to be isomorphic we must have:

- $v_1 = v_2$
- $e_1 = e_2$

In words, isomorphic graphs must have the same number of vertices and edges.

It is important to note that just having $v_1 = v_2$ and $e_1 = e_2$ is NOT a guarantee that two graphs will be isomorphic.
Degree of a Vertex

Definition (Degree of Vertex)

The **degree of a vertex** is the number of edges attached to that vertex.

Example (Degree of Each Vertex Shown)

![Graphs showing degree of each vertex](image)

We can use the idea of degree of a vertex to help us better understand when two graphs might be isomorphic.
Example (Isomorphic or Not Isomorphic 3)

- Draw the graph with vertices $A, B, C, D$ and edge set

  \{AB, AC, AD, BC, BD\}

- Is your graph isomorphic to one of the graphs below?

- Can you relabel the vertices of either pictured graph to match your graph?
Theorem (Isomorphic Graphs Theorem 2)

Suppose we have two graphs where each graph has the same number of vertices, \( v_1 = v_2 = n \). Write the degrees of each vertex (with repeats) in ascending order for Graph 1. This gives a list of numbers that we can represent generally as \( d_1, d_2, d_3, \ldots, d_n \).

If the two graphs are isomorphic then when listing the degrees of Graph 2 in ascending order, we get the exact same list as above.

In short, **ISOMORPHIC GRAPHS HAVE THE SAME DEGREE LISTS**.

More useful though, **IF THE DEGREE LISTS ARE DIFFERENT, THE TWO GRAPHS ARE NOT ISOMORPHIC**.

There are more sophisticated ways to determine if two graphs are isomorphic, but generally this is a VERY HARD question to resolve.
Consider the following graphs below:

- What is a reason(s) for why these graphs could be isomorphic?
- What is a reason(s) for why these graphs could *NOT* be isomorphic?
Consider the following graphs below:

What is a reason(s) for why these graphs could be isomorphic?

What is a reason(s) for why these graphs could NOT be isomorphic?
Graph Isomorphism

The “Determining If Two Graphs Are Isomorphic Theorems 1 & 2” are mostly useful for showing that two graphs are **NOT** isomorphic.

**Definition (Graph Isomorphism)**

If two graphs are isomorphic then there is a **Graph Isomorphism** that describes how they are the same. In practice this is:

- A relabeling of the vertices of Graph 1 so that each corresponds to the “same” vertex of Graph 2;

- This relabeling is done so that any edge of Graph 1 has a corresponding edge of Graph 2 under the new labels.

To determine a graph isomorphism, a really good place to start is to find the degrees of the vertices of **BOTH** graphs.
Step 1: List the degrees, in ascending order, of both graphs:

Left Graph: \( A, E, F, B, C, D \)
Right Graph: \( W, Y, Z, U, X, V \)
Graph Isomorphism Procedure: Step 2

Example (Describing a Graph Isomorphism)

- **Step 2: Understand the connections of the vertices:**
  - In the Left Graph, vertices A, E, and F all connect with B, C, and D. These are their only connections.
  - In the Right Graph, vertices W, Y, and Z all connect with U, V, and X. These are their only connections.
  - In the both graphs, Vertices of degree 4 connect with the one Vertex of Degree 5.
Step 3: Define the isomorphism:

- Because of the types of connections, any vertex of degree 3 in the Left Graph is like any vertex of degree 3 in the Right Graph. The same is true for the vertices of degree 4.

- \( A \leftrightarrow W, \; E \leftrightarrow Y, \; F \leftrightarrow Z, \; B \leftrightarrow U, \; C \leftrightarrow X, \; D \leftrightarrow V \).

- We can check by redrawing the Right Graph with vertices in the positions given by the isomorphism.
Consider the following graphs below:

What is a reason(s) for why these graphs could be isomorphic?

What is a reason(s) for why these graphs could NOT be isomorphic?
Example (Using Vertex Degrees 1)

- Make a graph with 5 vertices labeled with the letters A, B, C, D, and E.

- Include 8 edges connecting the vertices. You decide which connections to make!

- Is your graph isomorphic to this one?
Example (Using Vertex Degrees 2)

- Make another graph with 5 vertices labeled with the letters V, W, X, Y, and Z.
- Connect the vertices so that the degrees are 2, 3, 3, 4, 4
- Is your graph isomorphic to this one?
Example (Using Vertex Degrees 3)

- Are the two graphs below isomorphic?

- Are the two graphs below isomorphic?
Previously you made a graph with:

5 vertices labeled with the letters A, B, C, D, and E, connected by 8 edges.

- Compare your graph with a neighbor’s graph. Are your two graphs isomorphic?
  - If not isomorphic, how can you tell?
  - If they are isomorphic, find a correspondence between vertices.
Example (Compare With a Neighbor 2)

Previously you made a graph with:

5 vertices labeled with the letters V, W, X, Y, and Z, connected so the degrees are 2, 3, 3, 4, 4.

- Compare your graph with a neighbor’s graph. Are your two graphs isomorphic?
  - If not isomorphic, how can you tell?
  - If they are isomorphic, find a correspondence between vertices.
Theorem (Sum of Degrees of Vertices Theorem)

Suppose a graph has \( n \) vertices with degrees \( d_1, d_2, d_3, \ldots, d_n \).
Add together all degrees to get a new number
\[
d_1 + d_2 + d_3 + \ldots + d_n = D_v.
\]
Then \( D_v = 2e \).

In words, for any graph the sum of the degrees of the vertices equals twice the number of edges.

Stated in a slightly different way, \( D_v = 2e \) says that \( D_v \) is **ALWAYS** an even number.

Example (Using the Sum of Degrees of Vertices Theorem)

It is impossible to make a graph with \( v = 6 \) where the vertices have degrees 1, 2, 2, 3, 3, 4. This is because the sum of the degrees \( D_v \) is

\[
D_v = 1 + 2 + 2 + 3 + 3 + 4 = 15
\]

\( D_v \) is always an even number but 15 is odd!
Example (Using the Sum of the Degrees of Vertices Formula 1)

Consider the following scenarios:

- A graph has 4 vertices with degrees 0, 0, 0, and 0. What does this graph look like?

- A graph has 1 vertex with degree 2. What does this graph look like?

- A graph has 4 vertices with degrees 2, 3, 3, and 4. How many edges are there?

- A graph has 4 vertices with degrees 2, 2, 2, and 4. Can you say what this graph looks like?
Example (Using the Sum of the Degrees of Vertices Formula 2)

Consider the following scenarios:

- Is it possible to have a graph with vertices of degrees: 1 and 1?
- Is it possible to have a graph with vertices of degrees: 1 and 2?
- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3?
- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3, 3?