On the heat flow of equation of surfaces of constant mean curvature

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Abstract

We consider the initial and boundary value problem of heat flow of equation of surfaces of constant mean curvature. We give sufficient conditions on the initial data such that the heat flow develops finite time singularity. We also provide a new set of initial data to guarantee the existence of global regular solutions to the heat flow that converges to zero in H^1 exponentially as time goes to infinity.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Given a continuous function $H : \mathbb{R}^3 \to \mathbb{R}$, a map $u \in C^2(\Omega, \mathbb{R}^3)$ is called a H-surface, if it satisfies

$$\triangle u = 2H(u) u_x \wedge u_y, \quad \text{in } \Omega. \tag{1.1}$$

Here $\wedge$ denotes the wedge product of $\mathbb{R}^3$. In fact, if u is a conformal representation of a surface S in $\mathbb{R}^3$, i.e., $u_x \cdot u_y = 0 = |u_x|^2 - |u_y|^2$, then $H(u)$ is the mean curvature of S at the point u.

The boundary value problem for the equation of H-surface (1.1) with constant mean curvature H has been extensively studied by Wente [18], Hildebrandt [9], Struwe [16], and Brezis-Coron [1][2]. For variable H, there are recent works by Rey [12] and Caldiroli-Musina [3].

In this paper, we are interested in the initial-boundary value problem for the heat flow of the equation of H-surface:

$$\begin{cases} u_t = \triangle u - 2H(u) u_x \wedge u_y, & \text{in } \Omega \times (0, \infty), \\ u\big|_{t=0} = u_0, & \text{in } \Omega, \\ u\big|_{\partial\Omega} = \chi, & t > 0, \end{cases} \tag{1.2}$$

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where $u_0 \in H^1(\Omega)$, $\chi \in H^{\frac{3}{2}}(\partial\Omega)$, and $u_0|_{\partial\Omega} = \chi$. The equation $(1.2)_1 - (1.2)_2$ has been employed by Struwe [16] to obtain the existence of surfaces of constant mean curvatures $H = H_0$ with free boundaries under the condition $|H_0| \|u_0\|_{L^\infty} < 1$. Rey [12] extended the main result of [16] to variable H under the assumption

$$\|H\|_{L^\infty} \|u_0\|_{L^\infty} < 1. \tag{1.3}$$

For an arbitrary Lipschitz continuous function H, Wang [17] proved that if $u \in H^1(\Omega \times (0, +\infty))$ is a weak solution of $(1.2)_1$, then $u \in C^{2,\alpha}(\Omega \times (0, +\infty) \backslash \Sigma, \mathbb{R}^3)$, where $\Sigma = \bigcup\limits_{t>0} \Sigma_t \subset \Omega \times (0, +\infty)$ is a closed subset, whose Lebesgue measure is zero and $\Sigma_t \subset \Omega \times \{t\}$ is finite for almost all $t > 0$. Chen-Levine [6] has shown the existence and uniqueness of short time regular solution to (1.2) for $u_0 \in H^1(\Omega)$ and $\chi \in H^{\frac{3}{2}}(\partial\Omega)$.

We would like to remark that the structure of quadratic nonlinearity of (1.2) is similar to the structure of nonlinearity of the heat flow of harmonic maps in dimension two, which has been extensively studied by many people (see, for example, Eells-Sampson [7], Struwe [15], Chang [4], Qing [13], Qing-Tian [14], Lin-Wang [11] and Chang-Liu [5]). It is natural to utilize and extend the techniques in the study of heat flow of harmonic maps to study (1.2). However, there is an essential difference between the heat flow of harmonic maps and (1.2). Since the heat flow of harmonic maps is the gradient flow of the Dirichlet energy functional, which usually enjoys the energy inequality:

$$\int_\Omega |\nabla u|^2(\cdot, t) \le \int_\Omega |\nabla u|^2(\cdot, s), \quad 0 \le s \le t < \infty. \tag{1.4}$$

While even smooth solutions to (1.2) may not satisfy (1.4). This makes the study of certain analytic properties of (1.2) more subtle.

The aim of this paper is to address the existence of finite time singularities of (1.2) and new sufficient conditions on the initial data to assure the existence of global regular solutions to (1.2).

In order to describe our results, we need to recall a few notations. For any measurable set $D \subset \Omega$, define the Dirichlet energy of u on D by

$$E(u, D) = \frac{1}{2} \int_D |\nabla u|^2,$$

and write $E(u) = E(u, \Omega)$. Also, define

$$\mathcal{E}(u) = E(u) + V_H(u) = E(u) + \frac{2}{3} \int_\Omega H(u) u \cdot u_x \wedge u_y.$$

Define $B_R(z_0) = \{z \in \mathbb{R}^2 |\ |z - z_0| < R\}$ and write $B_R = B_R(0)$. Henceforth we assume

$$H = H_0 = \text{constant}, \text{ and } \chi = 0. \tag{1.5}$$

Then we have

Theorem 1.1 *Under the assumption (1.5) with $H_0 \neq 0$. If $0 \neq u_0 \in H_0^1(\Omega, \mathbb{R}^3)$, then the local regular solution u to (1.2) must blow up at finite time, provided that either*
(1) $\mathcal{E}(u_0) \leq 0$, or
(2) $0 < \mathcal{E}(u_0) < \frac{4\pi}{3H_0^2}$ and $\left|\int_\Omega u_0 \cdot u_{0x} \wedge u_{0y}\right| > \frac{4\pi}{|H_0|^3}$.

Theorem 1.2 *Under the assumption (1.5) with $H_0 \neq 0$. If $0 \neq u_0 \in H_0^1(\Omega, \mathbb{R}^3)$ satisfies*

$$0 < \mathcal{E}(u_0) < \frac{4\pi}{3H_0^2} \text{ and } \left|\int_\Omega u_0 \cdot u_{0x} \wedge u_{0y}\right| < \frac{4\pi}{|H_0|^3} \tag{1.6}$$

then there exists a unique global regular solution u to (1.2). Moreover, there exists $\alpha > 0$ such that

$$\max\{\|u(t)\|_2^2,\ \|\nabla u(t)\|_2^2\} = O(e^{-\alpha t}) \quad \textit{as } t \to \infty. \tag{1.7}$$

We will see from section 2 that there is no $u \in H_0^1(\Omega, \mathbb{R}^3)$ such that $\mathcal{E}(u) < \frac{4\pi}{3H_0^2}$ and $|\int_\Omega u_0 \cdot u_{0x} \wedge u_{0y}| = \frac{4\pi}{|H_0|^3}$. Moreover, if $|H_0|\|u_0\|_{L^\infty} < 1$, then either $\mathcal{E}(u_0) \geq \frac{4\pi}{3H_0^2}$ or (1.6) holds. It remains to be an interesting question to investigate (1.2) when the initial data u_0 has $\mathcal{E}(u_0) \geq \frac{4\pi}{3H_0^2}$.

The paper is written as follows. In section 2, we prove Theorem 1.1. In section 3, we prove Theorem 1.2.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, we recall the following isoperimetric inequality, whose proof can be found in [2] and [18].

Lemma 2.1 *(1). For any $u \in H_0^1(\Omega; \mathbb{R}^3)$, there holds*

$$\int_\Omega |\nabla u|^2 \geq \sqrt[3]{32\pi}\left(\int_\Omega u \cdot u_x \wedge u_y\right)^{2/3}. \tag{2.1}$$

(2). The best constant

$$\sqrt[3]{32\pi} = \inf_{\substack{u\in H_0^1(\Omega,\mathbb{R}^3)\\ \int_\Omega u\cdot u_x\wedge u_y=1}} \int_\Omega |\nabla u|^2,$$

can't be achieved whenever Ω is a bounded smooth domain.

Lemma 2.2 *If $u \in H_0^1(\Omega, \mathbb{R}^3)$ satisfies*

$$|\int_\Omega u \cdot u_x \wedge u_y| = \frac{4\pi}{|H_0|^3}, \tag{2.2}$$

then

$$\mathcal{E}(u) \geq \frac{4\pi}{3H_0^2}.$$

Proof. Applying the isoperimetric inequality (2.1), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\geq \sqrt[3]{32\pi}\left(\int_\Omega u\cdot u_x \wedge u_y\right)^{2/3}\\ &= \sqrt[3]{32\pi}(\frac{4\pi}{|H_0|^3})^{\frac{2}{3}} = \frac{8\pi}{H_0^2}.\end{aligned}$$

Hence

$$\mathcal{E}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{2|H_0|}{3}|\int_\Omega u\cdot u_x \wedge u_y| \geq \frac{4\pi}{3H_0^2}.$$

This completes the proof. □

We also need the following energy inequality for regular solutions to (1.2).

Lemma 2.3 *For $0 < T \leq \infty$, suppose that $u : \Omega \times [0, T) \to \mathbb{R}^3$ is a regular solution to (1.2). Then it holds*

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^2 + \mathcal{E}(u(t_2)) = \mathcal{E}(u(t_1)),\ \forall 0 \leq t_1 \leq t_2 < T. \tag{2.3}$$

Proof. Multiplying $(1.2)_1$ by u_t and integrating over Ω, applying integration by parts yields (2.3). □

Proof of Theorem 1.1: We argue by contradiction. Suppose that there exists a global regular solution $u \in C^\infty(\overline{\Omega} \times (0, +\infty), \mathbb{R}^3)$ to (1.2). Set

$$f(t) = \int_0^t \int_\Omega |u(t)|^2,\ t > 0.$$

Multiplying $(1.2)_1$ by u and integrating over $\Omega \times (0,t)$, we have

$$\int_\Omega |u(t)|^2 - \int_\Omega |u_0|^2 = -2\int_0^t\int_\Omega (|\nabla u|^2 + 2H_0 u\cdot u_x \wedge u_y).$$

By the definition of $f(t)$, we have $f'(t) = \int_\Omega |u(t)|^2$ and hence

$$f'(t) = \int_\Omega |u_0|^2 - 2\int_0^t\int_\Omega (|\nabla u|^2 + 2H_0 u\cdot u_x \wedge u_y), \tag{2.4}$$

and

$$f''(t) = -2\int_\Omega (|\nabla u|^2 + 2H_0 u\cdot u_x \wedge u_y)(t). \tag{2.5}$$

Since

$$2H_0\int_\Omega u\cdot u_x \wedge u_y(t) = 3(\mathcal{E}(u(t)) - \frac{1}{2}\int_\Omega |\nabla u(t)|^2),$$

(2.5) and (2.3) give

$$f''(t) = \int_\Omega |\nabla u(t)|^2 - 6\mathcal{E}(u(t)) = [\int_\Omega |\nabla u(t)|^2 - 6\mathcal{E}(u_0)] + 6\int_0^t\int_\Omega |u_t|^2. \tag{2.6}$$

Now we claim

$$\int_\Omega |\nabla u(t)|^2 - 6\mathcal{E}(u_0) \geq 0,\ t > 0. \tag{2.7}$$

Assume this claim for the moment. Then we obtain

$$f''(t) \geq 6\int_0^t\int_\Omega |u_t|^2. \tag{2.8}$$

Now we need to show

$$\int_0^1\int_\Omega |u_t|^2 > 0. \tag{2.9}$$

For, otherwise, u_0 solves

$$\Delta u_0 = 2H_0 u_{0x} \wedge u_{0y}$$

so that, by multiplying the equation by u_0 and integrating over Ω,

$$\int_\Omega |\nabla u_0|^2 + 2H_0\int_\Omega u_0\cdot u_{0x} \wedge u_{0y} = 0. \tag{2.10}$$

This implies

$$\mathcal{E}(u_0) = \frac{1}{6}\int_\Omega |\nabla u_0|^2 > 0.$$

Hence u_0 satisfies the condition (2), i.e. $\mathcal{E}(u_0) < \frac{4\pi}{3H_0^2}$. Thus we have

$$\int_\Omega |\nabla u_0|^2 < \frac{8\pi}{H_0^2}. \tag{2.11}$$

On the other hand, the condition (2) gives

$$\left| \int_\Omega u_0 \cdot u_{0x} \wedge u_{0y} \right| \geq \frac{4\pi}{|H_0|^3} \tag{2.12}$$

It is clear that (2.11) and (2.12) contradicts (2.10). It follows from (2.9) that $f(t)$ is strictly convex for $t \geq 1$, i.e.,

$$f''(t) \geq 6 \int_0^1 \int_\Omega |u_t|^2 > 0, \ \forall t \geq 1.$$

Thus we have

$$\lim_{t \to +\infty} f(t) = \lim_{t \to +\infty} f'(t) = +\infty. \tag{2.13}$$

However we have, for $t \geq 1$,

$$\begin{aligned} f(t)f''(t) \geq & 6 \left(\int_0^t \int_\Omega |u|^2 \right) \cdot \left(\int_0^t \int_\Omega |u_t|^2 \right) \\ \geq & 6 \left(\int_0^t \int_\Omega u u_t \right)^2 = \frac{3}{2} \left(f'(t) - f'(0) \right)^2. \end{aligned}$$

This, combined with (2.13), implies that there is a $\alpha \in (0, \frac{1}{2})$ such that for any sufficiently large time t,

$$f(t)f''(t) \geq (1+\alpha)(f'(t))^2. \tag{2.14}$$

Hence $f(t)^{-\alpha}$ is strictly concave for sufficiently large time t. This contradicts the fact that

$$f(t)^{-\alpha} > 0, \quad \lim_{t \to +\infty} f(t)^{-\alpha} = 0.$$

Thus the short time regular solution u to (1.2) must blow up at finite time.

Now we return to the proof of (2.7). It is obvious that (2.7) holds if u_0 satisfies the condition (1). It remains to verify (2.7) when u_0 satisfies the condition (2). Since $\mathcal{E}(u_0) < \frac{4\pi}{3H_0^2}$, Lemma 2.3 implies

$$\mathcal{E}(u(t)) < \frac{4\pi}{3H_0^2}, \ \forall t > 0. \tag{2.15}$$

Now we claim

$$\left|\int_\Omega u\cdot u_x\wedge u_y\right|(t) > \frac{4\pi}{|H_0|^3},\ \forall t\geq 0. \tag{2.16}$$

Notice (2.16) holds for $t=0$. If (2.16) were false, then there would exist $t_0>0$ such that

$$\left|\int_\Omega u\cdot u_x\wedge u_y\right|(t_0) = \frac{4\pi}{|H_0|^3}. \tag{2.17}$$

But (2.17) and (2.15) would contradict Lemma 2.2. Hence (2.16) holds.

It follows from (2.16) and the isoperimetric inequality that we have

$$\int_\Omega |\nabla u(t)|^2 \geq \frac{8\pi}{H_0^2},\ \forall t>0.$$

Hence

$$\int_\Omega |\nabla u(t)|^2 - 6\mathcal{E}(u_0) \geq \frac{8\pi}{H_0^2} - 6(\frac{4\pi}{3H_0^2}) = 0.$$

This proves (2.7). Hence the proof of Theorem 1.1 is complete. □

3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. First we will show that the short time regular solution can be extended to be global regular solution. In order to do it, we perform the blow-up analysis to rule out any possible finite singular time. Then we derive the exponential convergence at time infinity.

Proof of Theorem 1.2. Suppose that the short time regular solution develops a finite time singularity. Let $0<T^*=T^*_{max}$ be the maximal time interval such that there exists a regular solution $u\in C^\infty(\overline{\Omega}\times(0,T^*),\mathbb{R}^3)$ to (1.2). Then $T^*<+\infty$. It follows from Lemma 2.3 that there is a $\delta_0>0$ such that

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u_0) \leq \frac{4\pi}{3H_0^2} - \delta_0,\ 0\leq t<T^*. \tag{3.1}$$

We now claim

$$\left|\int_\Omega u\cdot u_x\wedge u_y\right|(t) < \frac{4\pi}{|H_0|^3},\ \forall 0\leq t<T^*. \tag{3.2}$$

Notice (3.2) holds for $t=0$. If (3.2) were false, then there exists $0<t_1<T^*$ such that $u_2=u(t_1)$ satisfies

$$\left|\int_\Omega u_2\cdot u_{2x}\wedge u_{2y}\right| = \frac{4\pi}{|H_0|^3}. \tag{3.3}$$

But (3.3) and (3.2) would contradict Lemma 2.2.

It follows from (3.2) and (2.3) that

$$\int_\Omega |\nabla u(t)|^2 \le \frac{8\pi}{H_0^2} - \delta_0,\ \forall 0 \le t < T^*. \tag{3.4}$$

$$\int_0^{T^*} \int_\Omega |u_t|^2 \le \frac{4\pi}{3H_0^2}. \tag{3.5}$$

It follows from [6] Theorem 5.1 that T^* can be characterized by the fact that there exists a universal constant $\epsilon_0 > 0$ such that

$$\limsup_{t \nearrow T^*} \max_{z \in \overline{\Omega}} E(u(t); \Omega \cap B_R(z)) \ge \epsilon_0^2,\ \forall R > 0. \tag{3.6}$$

(3.6) implies that for any $0 < \epsilon_1 < \epsilon_0$, there exist $0 < t_0 < T^*$, $r_n \downarrow 0$, and $t_n \uparrow T^*$ such that

$$\epsilon_1^2 = \max_{z \in \overline{\Omega}, t_0 \le t \le t_n} E(u(t); \Omega \cap B_{r_n}(z)). \tag{3.7}$$

If $\epsilon_1 > 0$ is sufficiently small, then (3.7) and the local energy inequality (see [6] Lemma 4.5) imply that there exists $\theta_0 \in (0,1)$, depending only on ϵ_1 and $\mathcal{E}(u_0)$, and $z_n \in \Omega$ such that

$$\int_{\Omega \cap B_{2r_n}(z_n)} |\nabla u|^2(t_n - \theta_0 r_n^2) \quad \ge \quad \frac{1}{2} \max_{z \in \overline{\Omega}} \int_{\Omega \cap B_{2r_n}(z)} |\nabla u|^2(t_n - \theta_0 r_n^2) \ge \frac{\epsilon_1^2}{4}.$$

Set $\Omega_n = r_n^{-1}(\Omega \setminus \{z_n\})$. Define $v_n : \Omega_n \times [\frac{t_0 - t_n}{r_n^2}, 0] \to \mathbb{R}^3$ by

$$v_n(z,t) = u(z_n + r_n, t_n + r_n^2 t).$$

Then v_n solves (1.2) on $\Omega_n \times [\frac{t_0 - t_n}{r_n^2}, 0]$. Moreover,

$$\int_{\Omega_n \cap B_2(0)} |\nabla v_n|^2(-\theta_0) \ge \frac{\epsilon_1^2}{4}, \tag{3.8}$$

$$\max_{(z,t) \in \Omega_n \times [\frac{t_0 - t_n}{r_n^2}, 0]} \int_{\Omega_n \cap B_1(z)} |\nabla v_n(t)|^2 \le \epsilon_1^2, \tag{3.9}$$

and for any fixed $T > 0$,

$$\int_{-T}^0 \int_{\Omega_n} |v_{nt}|^2 = \int_{t_n - T r_n^2}^{t_n} \int_\Omega |u_t|^2 \to 0,\ \text{as } n \to +\infty. \tag{3.10}$$

It follows from (3.9) and the small energy regularity theorem (see [6] Lemma 4.6) that for any $T > 0$,

$$\|v_n\|_{C^k((\Omega_n \cap B_1(z))\times[-T,0])} \leq C(k, T, \epsilon_1), \ \forall k \geq 1, \ z \in \Omega_n. \tag{3.11}$$

We may assume $z_n \to z_0 \in \overline{\Omega}$. We divide the proof into two cases.
Case 1. $z_0 \in \Omega$. Then it is clear that

$$\frac{dist(z_n, \partial\Omega)}{r_n} \to +\infty \ \text{ and } \ \Omega_n \to \mathbb{R}^2.$$

Moreover, by (3.9), (3.10), and (3.11), we may assume that

$$v_n \to \omega \text{ strongly in } H^1_{\text{loc}} \cap C^2_{\text{loc}}(\mathbb{R}^2 \times (-\infty, 0], \mathbb{R}^3).$$

It is clear that

$$\omega_t \equiv 0 \text{ on } \mathbb{R}^2 \times (-\infty, 0], \tag{3.12}$$

$$\int_{B_2} |\nabla\omega|^2(-\theta_0) \geq \frac{\epsilon_1^2}{4}, \tag{3.13}$$

and for any $R > 0$,

$$\begin{aligned}
\int_{B_R} |\nabla\omega|^2(-\theta_0) &= \lim_{n\to+\infty} \int_{B_R} |\nabla v_n|^2 \\
&= \lim_{n\to+\infty} \int_{\Omega\cap B_{Rr_n}(z_n)} |\nabla u|^2(t_n - \theta_0 r_n^2) \\
&\leq \frac{8\pi}{H_0^2} - \delta_0.
\end{aligned} \tag{3.14}$$

It follows from (3.12), (3.13), and (3.14) that $\omega \in H^1 \cap C^\infty(\mathbb{R}^2, \mathbb{R}^3)$ is a nontrivial solution to

$$\triangle\omega = 2H_0\omega_x \wedge \omega_y \ \text{ in } \mathbb{R}^2, \tag{3.15}$$

and satisfies

$$\int_{\mathbb{R}^2} |\nabla\omega|^2 \leq \frac{8\pi}{H_0^2} - \delta_0. \tag{3.16}$$

On the other hand, a well-known theorem of Brezis-Coron (see [1] Lemma A.1) asserts that any nontrivial solution ω of (3.15) must have

$$\int_{\Omega} |\nabla\omega|^2 \geq \frac{8\pi}{H_0^2}.$$

This contradicts (3.16).

Case 2. $z_0 \in \partial\Omega$. In this case, we have either

(2a) $\lim_{n\to\infty} \frac{\mathrm{dist(z_n,\partial\Omega)}}{r_n} = +\infty$, or

(2b) $\lim_{n\to\infty} \frac{\mathrm{dist(z_n,\partial\Omega)}}{r_n} = L < +\infty$.

It is not hard to see that the same argument as Case 1 shows (2a) can't happen. Hence we only need to consider (2b). For simplicity, we may assume $L = 0$. Hence $\Omega_n \to \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$. Since $v_n|_{\partial\Omega_n} = 0$, we see that $v_n \to \omega$ strongly in $H^1 \cap C^2(B^+_R \times [-R^2, 0])$ for any $R > 0$, where $B^+_R = B_R \cap \mathbb{R}^2_+$. Moreover,

$$\omega_t \equiv 0 \text{ on } \mathbb{R}^2_+ \times (-\infty, 0], \tag{3.17}$$

$$0 < \int_{\mathbb{R}^2_+} |\nabla\omega|^2 < \frac{8\pi}{H_0^2}, \tag{3.18}$$

and

$$\Delta\omega = 2H_0\omega_x \wedge \omega_y \text{ in } \mathbb{R}^2_+;\ \omega\big|_{\partial\mathbb{R}^2_+} = 0. \tag{3.19}$$

It is a well-known fact that any H^1 solution ω to (3.19) is zero. Here we provide a simple proof (see also [1] Lemma A.1). First, let $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^3$ be the odd extension of ω with respect to y, i.e.

$$\hat{\omega}(x,y) = \omega(x,y) \text{ for } y \geq 0; = -\omega(x,-y) \text{ for } y \leq 0.$$

Then it is easy to verify that $\hat{\omega} \in H^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ also solves (3.15). Consider the Hopf differential $\mathcal{H}(\hat{\omega}) = |\hat{\omega}_x|^2 - |\hat{\omega}_y|^2 - 2i\hat{\omega}_x \cdot \hat{\omega}_y$ of $\hat{\omega}$. Then one can check that $\mathcal{H}(\hat{\omega})$ is holomorphic, i.e.,

$$\frac{\partial\mathcal{H}(\omega)}{\partial\bar{z}} = 0 \text{ in } \mathbb{R}^2.$$

Since $\mathcal{H}(\hat{\omega}) \in L^1(\mathbb{R}^2)$, we conclude that $\mathcal{H}(\hat{\omega}) \equiv 0$ in $\mathbb{R}^2$. In particular, ω is conformal in $\mathbb{R}^2_+$. Since $\omega\big|_{\partial\mathbb{R}^2_+} = 0$, $\omega_x \equiv 0$ on $\partial\mathbb{R}^2$. Hence $\omega_y \equiv 0$ on $\mathbb{R}^2_+$. This implies that $\omega \equiv 0$ in $\mathbb{R}^2_+$. This contradicts (3.18).

It is now evident that $T^* = +\infty$, i.e. the local regular solution u is a global regular solution. Finally we want to prove (1.7). Set

$$F(u) = \int_\Omega |\nabla u|^2 + \int_\Omega 2H_0 u \cdot u_x \wedge u_y.$$

First observe that (3.2) and the isoperimetric inequality (2.1) imply that

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\geq \left(\frac{32\pi}{|\int_\Omega u\cdot u_x\wedge u_y|}\right)^{\frac{1}{3}}\left|\int_\Omega u\cdot u_x\wedge u_y\right| \\ &= \sqrt[3]{8|H_0|^3}\left|\int_\Omega u\cdot u_x\wedge u_y\right| = 2|H_0\int_\Omega u\cdot u_x\wedge u_y|. \quad (3.20)\end{aligned}$$

Hence

$$\mathcal{E}(u(t)) \geq \frac{1}{6}\int_\Omega |\nabla u(t)|^2,\ \forall t>0. \quad (3.21)$$

The isoperimetric inequality (2.1) implies that for any $t>0$,

$$\begin{aligned}\left|\int_\Omega 2H_0 u(t)\cdot u_x(t)\wedge u_y(t)\right| &\leq \left(\frac{H_0^2}{8\pi}\int_\Omega |\nabla u(t)|^2\right)^{\frac{1}{2}}\int_\Omega |\nabla u(t)|^2 \\ &\leq \left(\frac{3H_0^2}{4\pi}\mathcal{E}(u(t))\right)^{\frac{1}{2}}\int_\Omega |\nabla u(t)|^2 \\ &\leq \left(\frac{3H_0^2}{4\pi}\mathcal{E}(u_0)\right)^{\frac{1}{2}}\int_\Omega |\nabla u(t)|^2,\end{aligned}$$

where we have used both (3.21) and (2.3) in the last two inequalities.

Since $\mathcal{E}(u_0) < \frac{4\pi}{3H_0^2}$, we have

$$0<\delta\equiv\left(\frac{3H_0^2\mathcal{E}(u_0)}{4\pi}\right)^{1/2}<1.$$

Let $\gamma = 1-\delta>0$. Then we have, for any $t>0$,

$$\left|\int_\Omega 2H_0 u(t)\cdot u_x(t)\wedge u_y(t)\right| \leq (1-\gamma)\int_\Omega |\nabla u(t)|^2. \quad (3.22)$$

Hence

$$F(u(t)) \geq \gamma\int_\Omega |\nabla u(t)|^2,\ \forall t>0. \quad (3.23)$$

Direct calculations imply that

$$\frac{d}{dt}\int_\Omega |u(t)|^2 = -2F(u(t)),\ \forall t>0.$$

Integrating this inequality over $[t,+\infty)$ yields

$$\int_t^{+\infty} F(u(\tau))\,d\tau \leq \frac{1}{2}\int_\Omega |u(t)|^2 \leq C\int_\Omega |\nabla u(t)|^2,\ \forall t>0, \quad (3.24)$$

where we have used the Poincaré inequality in the last inequality. Combining (3.23) with (3.24), we obtain

$$\int_t^{+\infty}\int_\Omega |\nabla u|^2 \le C\int_\Omega |\nabla u(t)|^2,\ \forall t>0. \tag{3.25}$$

Set $G(t)=\int_t^{+\infty}\int_\Omega |\nabla u|^2$. Then (3.25) becomes

$$CG'(t)+G(t)\le 0,\ \forall t>0. \tag{3.26}$$

Hence

$$G(t)\le G(0)e^{-\frac{t}{C}},\ \forall t>0. \tag{3.27}$$

Since (3.20) also implies

$$\mathcal{E}(u(t))\le \frac{5}{6}\int_\Omega |\nabla u(t)|^2,\ \forall t>0,$$

(3.27) gives

$$\int_t^{+\infty}\mathcal{E}(u(\tau))\,d\tau \le C_0e^{-\frac{t}{C}},\ \forall t>0. \tag{3.28}$$

Since $\mathcal{E}(u(t))$ is monotone decreasing, this implies

$$\mathcal{E}(u(t+1))\le \int_t^{t+1}\mathcal{E}(u(\tau))\,d\tau \le C_0e^{-\frac{t}{C}},\ \forall t>0.$$

This, combined with the Poincaré inequality, implies (1.7). The proof of Theorem 1.2 is now complete. □

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