QUANTITATIVE LIOUVILLE THEOREMS

Rigidity theorems for mappings whose gradient lie in a subset of the conformal matrices date back to Liouville [Lio 50] who proved that given a domain $\Omega \subset \mathbb{R}^n$ and a function $u \in C^3\left(\Omega, \mathbb{R}^3\right)$ with the property that $Du\,(x) = \lambda\,(x)\,O\,(x)$ where $\lambda\,(x) \in \mathbb{R}_+$ and $O\,(n) \in SO\,(n)$ then u is either affine or a Mobius transformation. Generalisations of this theorem have been provided by Gehring, Reshetnyak, Bojarski and Iwaniec.

A corollary to Liouville's Theorem is that a C^3 function whose gradient belongs everywhere to $SO\,(n)$ is an affine mapping. A striking quantitative version of this fact was recently proved by Friesecke, James and Müller [Fr-Ja-Mu 02], who showed that for every bounded open connected Lipschitz domain $U \subset \mathbb{R}^n$, $n \geq 2$, and every $q > 1$, there exists a constant $C(U, q)$ such that, writing $K := SO(n)$,

$$\inf_{R \in K} \|Dv - R\|_{L^q(U)} \leq C\,(U, q)\,\|d\,(Dv, K)\,\|_{L^q(U)} \qquad \text{for every } v \in W^{1,q}(U; \mathbb{R}^n).$$

Here and below, $d(M, K)$ denotes the distance from a matrix $M \in \mathbb{R}^{n \times n}$ to a subset $K \subset \mathbb{R}^{n \times n}$, measured in the Euclidean norm. This result strengthens earlier work of a series of authors, including John [Jo 61],[Jo 61], Reshetnyak [Re 67], and Kohn [Ko 82], and it has had a number of important applications. For example, it is a main tool used to provide a relatively complete analysis of the gamma limit of thin elastic structures, [Fr-Ja-Mu 02], [Fr-Ja-Mu 06].

The Friesecke, James, Müller result was a major improvement on previous theorems all of whom were sub-optimal in terms of control of the gradient of the function. The reason for the improvement was the introduction of regularity methods for elliptic PDE into the problem. The main part of the talk will be about the proof by Friesecke, James, Müller, specifically why it always such strong improvement on the previous 'geometrical methods'.

In the final part of the talk we will touch on a line of generalisation which has been contributed to by Sergio Conti, Ben Schweizer, Milena Chermisi, Robert Jerrard and the speaker:

If we consider two compatible wells $K = SO\,(n)\,A \cup SO\,(n)\,B$, i.e. wells for which there exists matrices $X \in SO\,(n)\,A$, $Y \in SO\,(n)\,B$ with $\mathrm{rank}\,(X - Y) = 1$, then the example of a piecewise affine function u such that $\mathrm{Image}(Du) = \{X, Y\}$ shows that no exact analog of (1) can hold. However, that a sort of 2-well theorem can hold provided one has suitable 'weak' control over second derivatives. One of the most recent results in this in this line of generalization is:

Theorem 1 (Jerrard-Lorent, 2008)**.** *Let $q \geq 1$, $p > 1$, A, B be $n \times n$ matrices with non-zero determinant. Define $K := SO(n)A \cup SO(n)B$. Let $\Omega \subset \mathbb{R}^n$ be a connected open Lipschitz domain and $\Omega' \subset\subset \Omega$ a subdomain that is strictly contained inside Ω. There exists positive constant $a = a\,(K, \Omega, \Omega') < 1$ such that for any $u \in W^{1,1} \cap W^{2,q}$, $u : \Omega \to \mathbb{R}^n$ that satisfies*

$$\int_\Omega d^p\,(Du, K)\,dx \leq a\varepsilon$$

and

$$\int_\Omega \left|D^2 u\right|^q dx \leq a\varepsilon^{1-q}$$

then there exists a constant $M = M\,(\Omega, \Omega', K, n, q, p) > 0$ and $M \in K$ such that

$$\int_{\Omega'} |Du - K|^p\,dx \leq M\varepsilon^{\frac{1}{p}}.$$

References

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